SCIENCE CLINIC

MATHS ESSENTIALS

GRADE

10

Content Acknowledgement

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Algebraic Expressions

TERMINOLOGY:

Numerical Coefficient: the number in front of a variable.

Variable: an algebraic letter(s) used to represent unknown(s).

Constant: the numerical term

Algebraic Expression: a mathematical expression made up of one or more terms separated by addition (+) or subtraction (–).

Polynomial: an algebraic expression where the exponent(s) on the variable(s) are natural numbers.
Monomial e.g. 4 or $2a^2bc$ (one term)
Binomial e.g. $6x+2y$ (two terms)
Trinomial e.g. $6x^2-5x+4$ (three terms)

Degree: is the highest value of an exponent of a specific variable in an algebraic expression.
(e.g. $7x^3-3xy+8x^6+4$ has the sixth degree in x and first degree in y)

Like Terms: are terms with the same variable(s) with the same exponents, the coefficients may differ.
(e.g. $6a^2b$ and $-\frac{1}{2}a^2b$)

Unlike Terms: are terms where the variables are different.
(e.g. $2x$, $2x^2$ and $3xy$)

SIMPLIFYING ALGEBRAIC EXPRESSIONS

Follow BODMAS rule but can only add or subtract like terms and write answer with variables in alphabetical order and terms in descending order of powers.

EXAMPLE
Simplify the following:

1. $6bca-7abc+4a^2bc-3cab+bca^2$
$=bca^2+4a^2bc+6bca-7abc-3cab$
$=5a^2bc-4abc$

2. $6x-4x^2-8x+x^3-x^2+7x-3x^3$
$=-3x^3+x^3-4x^2-x^2+6x-8x+7x$
$=-2x^3-5x^2+5x$

PRODUCTS

1. Monomial by a Polynomial

Use the distributive law

EXAMPLE
$2a^2(3a^2+4ab-a^3c)$
$=(2a^2\times 3a^2)+(2a^2\times 4ab)+(2a^2\times -a^3c)$
$=6a^4+8a^3b-2a^5c$
$=-2a^5c+6a^4+8a^3b$

2. Binomial by Binomial

Use FOIL method (Firsts, Outers, Inners, Lasts)

EXAMPLE
$(a-b)(x+y)$
$=(a\times x)+(a\times y)+(-b\times x)+(-b\times y)$
F O I L
$=ax+ay-bx-by$

EXAMPLE
$(2x+y)(3x-4y)$
$=(2x\times 3x)+(2x\times -4y)+(y\times 3x)+(y\times -4y)$
$=6x^2-8xy+3xy-4y^2$ (add like terms)
$=6x^2-5xy-4y^2$

3. Squaring a Binomial

Step 1: Square the first term
Step 2: Multiple the first term by the second term and double it
Step 3: Square the last term.

EXAMPLE
$(p+2r)^2$
$=(p)^2+(p\times 2r)\times 2+(2r)^2$
$=p^2+4pr+4r^2$

EXAMPLE
$(3a-4b)^2$
$=(3a)^2+(3a\times -4b)\times 2+(-4b)^2$
$=9a^2-24ab+16b^2$

NOTE: the second step in these examples is not usually shown.

4. Difference of Two Squares

The two binomials are the same except the sign in one is a plus and in the other is a minus. The outer and inners are then additive inverses of each other so answer is only first squared minus last squared.

EXAMPLE
$(3a-2b)(3a+2b)$
$=(3a)^2-(2b)^2$
$=9a^2-4b^2$

EXAMPLE
$(5x-7y^2)(5x+7y^2)$
$=25x^2-49y^4$

EXAMPLE
$[(a-b)+5][(a-b)-5]$
$=(a-b)^2-25$
$=a^2-2ab+b^2-25$

5. Binomial by a Trinomial

Multiply each term in the binomial by each term in the trinomial and the add like terms.

EXAMPLE
$(a-2)(a^2-a+1)$
$=a(a^2)+a(-a)+a(1)-2(a^2)-2(-a)-2(1)$
$=a^3-a^2+a-2a^2+2a-2$
$=a^3-3a^2+3a-2$

EXAMPLE
$(p+q)(p^2-pq+q^2)$
$=p^3-p^2q+pq^2+p^2q-pq^2+q^3$
$=p^3+q^3$ (sum of 2 cubes)

EXAMPLE
$(3a-2b)(9a^2+6ab+4b^2)$
$=27a^3+18a^2b+12ab^2-18a^2b-12ab^2-8b^3$
$=27a^3-8b^3$ (difference of 2 cubes)

6. Mixed Questions

Follow BODMAS;
1. Simplify in brackets if possible
2. Square binomial, FOIL or binomial by Trinomial
3. Distribution
4. Add or subtract like terms.

EXAMPLE
$4x(4xy-16y+12)-(2x+y)(x-y)$
$=4x(4xy-16y+12)-(2x^2-2xy+xy-y^2)$
$=16x^2y-64xy+48x-2x^2+2xy-xy+y^2$
$=-2x^2+16x^2y+48x-63xy+y^2$

EXAMPLE
$=3(a^2+3a-10)-2(a^2-6a+9)+2(a^2-4)$
$=3a^2+9a-30-2a^2+12a-18+2a^2-8$
$=3a^2+21a-56$

FACTORISATION

Factorisation is the inverse operation to products, that is we want to put the brackets back into the sum.

STEPS:

1. Look for a common factor first.
2. If a binomial look for difference of two squares or sum/ difference of two cubes.
3. If a trinomial check if in form $ax^2 - bx + c$, then factorise.
4. If 4 or more terms group by looking for patterns first, e.g. difference of squares or perfect square trinomial.
5. Don't forget to factorise as far as possible.
6. Remember terms in brackets can be considered as a variable,

1. Highest Common Factor (HCF):

This is inverse of distribution.

EXAMPLES

Factorise Fully:

1. $6y^2 + 12y$
   $= 6y(y + 2)$

2. $3a(2a - b) - a^2(2a - b)$ — Take $(2a - b)$ out as a common bracket
   $= a(2a - b)(3 - a)$

3. $x(x - y) - 4(x - y)^2$ — Take $(x - y)$ out as a common bracket

4. $= (x - y)[x - 4(x - y)]$ — Simplify 2nd bracket
   $= (x - y)[x - 4x + 4y]$
   $= (x - y)(-3x + 4y)$
   $= -(x - y)(3x - 4y)$

2. Sign change:

Change of sign in a bracket to make the factors the same.

NOTE:

$(b + a) = (a + b)$ **but** $(b - a) \neq (a - b)$

Do a sign change as follows:

$(b - a) = -1(a - b)$

EXAMPLE

$4a(a - 2b) - 6(2b - a)$
$= 4a(a - 2b) + 6(a - 2b)$
$= 2(a - 2b)(2a + 3)$

3. Difference of Two Squares (DOTS):

Square root the first term minus square root second term in one bracket then Square root the first term plus square root second term in second bracket.

EXAMPLES

Factorise Fully:

1. $4a^2 - 64b^2$ — Remember to check for HCF 1st
   $= 4(a^2 - 16b^2)$
   $= 4(a - 4b)(a + 4b)$

2. $x^2(x - k) + y^2(k - x)$ — Sign change
   $= x^2(x - k) - y^2(x - k)$ — Take $(x - k)$ out as a common bracket
   $= (x - k)(x^2 - y^2)$ — 2nd bracket is DOTS
   $= (x - k)(x - y)(x + y)$

3. $(a - b)^2 - (2a + b)^2$
   $= [(a - b) - (2a + b)][(a - b) + (2a + b)]$
   $= [a - b - 2a - b][a - b + 2a + b]$
   $= (-a - 2b)(3a)$
   $= -3a(a + 2b)$

4. Grouping:

Used if four or more terms. First group then do HCF. Groups can be due to HCF, Difference of two squares or perfect square trinomial.

EXAMPLES

Factorise Fully:

1. $9d + bc - bd - 9c$
   $= (9d - 9c) + (bc - bd)$ — HCF in each bracket
   $= 9(d - c) + b(c - d)$ — Sign change needed
   $= 9(d - c) - b(d - c)$ — Do HCF
   $= (d - c)(9 - b)$

2. $2a - 3b + 4a^2 - 9b^2$
   $= (2a - 3b) + (4a^2 - 3b^2)$ — 2nd bracket DOTS
   $= (2a - 3b) + (2a - 3b)(2a + 3b)$ — Do HCF
   $= (2a - 3b)(1 + 2a + 3b)$

3. $25a^2 - p^2 - 12pq - 36q^2$ — Group last three terms as they make a perfect square trinomial
   $= 25a^2 - (p^2 + 12pq + 36q^2)$
   $= 25a^2 - (p + 6q)^2$ — DOTS
   $= [5a - (p + 6q)][5a + (p + 6q)]$
   $= (5a - p - 6q)(5a + p + 6q)$

5. Trinomials:

STEPS:

1. Put in standard form $ax^2 + bx + c$
2. Multiply the coefficients of the 1st and 3rd terms (i.e. $a \times c$)
3. Find the factors of answer in (2) that add if $+c$ or subtract if $-c$ to get b
4. Write with middle term split into outers and inners
5. Factorise by grouping

EXAMPLES

Factorise Fully:

1. $3x^2 + 7x + 2$
   $= 3x^2 + 6x + 1x + 2$
   $= 3x(x + 2) + (x + 2)$
   $= (x + 2)(3x + 1)$

   $a \times c = 3 \times 2 = 6$
   $c = +2$; **+** factors;
   ~~2×3 ; $2 + 3 = 5$~~
   1×6 ; $1 + 6 = 7$

   both terms **+** as middle term is **+**

2. $6a^2 - 17ab + 12b^2$
   $= 6a^2 - 9ab - 8ab + 12b^2$
   $= 3a(2a - 3b) - 4b(2a - 3b)$
   $= (2a - 3b)(3a - 4b)$

   $a \times c = 6 \times 12 = 72$
   $c = +12$; **+** factors;
   9×8 ; $9 + 8 = 17$

   both terms **–** as middle term is **–**

3. $3p^2 + 7p - 6$
   $= 3p^2 + 9p - 2p - 6$
   $= 3p(p + 3) - 2(p + 3)$
   $= (p + 3)(3p + 2)$

   $a \times c = 3 \times 6 = 18$
   $c = -6$; **–** factors;
   9×2 ; $9 - 2 = 7$

   biggest factor gets middle term sign

4. $2x^2 - 6x - 36$
   $= 2(x^2 - 3x - 18)$ (HCF)
   $= 2(x^2 - 6x + 3x - 18)$
   $= 2[x(x - 6) + 3(x - 6)]$
   $= 2(x - 6)(x + 6)$

   $a \times c = 1 \times 18 = 18$
   $c = -3$; **–** factors;
   6×3 ; $3 - 6 = -3$

   biggest factor gets middle term sign

5. Perfect Square Trinomial

   a) $4m^2 - 18mn + 9n^2$
   $= \left(\sqrt{4m^2} - \sqrt{9n^2}\right)^2$
   $= (2m - 3n)^2$

   First term and last term are perfect squares

   b) $49p^4 + 84p^2 + 36$
   $= (7p^2 + 6)^2$

ALGEBRAIC EXPRESSIONS

FACTORISATION (CONTINUED)

6. Sum or Difference of Two Cubes:

STEPS:

Example: $8a^3 + 27$

1. First bracket (binomial):
   a. Cube root the 2 terms sign between that of sum
   $\left(\sqrt[3]{8a^3} + \sqrt[3]{27}\right)$
   $= (\mathbf{2a + 3})$
2. Second bracket (trinomial):
   a. square first term
   b. 1st term x 2nd term with opposite sign to 1st term
   c. add 2nd term squared
   $(2a)^2 - (2a \times 3) + (3)^2$
   2.a. 2.b. 2.c.
   $= (\mathbf{4a^2 - 6a + 9})$

$8a^3 + 27 = (2a + 3)(4a^2 - 6a + 9)$

EXAMPLES:

1. $8h^3 - 125g^3$
$= (2h - 5g)(4h^2 + 10gh + 25g^2)$

2. $24t^3 + 1029$
$= 3(8t^3 + 343)$
$= 3(2t + 7)(4t^2 - 14t + 49)$

3. $a^3 - \dfrac{216}{a^3}$
$= \left(a - \dfrac{6}{a}\right)\left(a^2 + 6 + \dfrac{36}{a^2}\right)$

1. Simplification of a Fraction with multiplication and division.

STEPS:

1. Factorise the numerator(s) and the denominator(s).
2. Cancel like factors.

REMEMBER: $\dfrac{a}{b} \div \dfrac{b}{a}$

$= \dfrac{a}{b} \times \dfrac{a}{b}$

EXAMPLES:

1. $\dfrac{x^2 - x - 6}{x^2 - 9}$

$= \dfrac{(x-3)(x+2)}{(x-3)(x+3)}$

$= \dfrac{\cancel{(x-3)}(x+2)}{\cancel{(x-3)}(x+3)}$

$= \dfrac{(x+2)}{(x+3)}$

2. $\dfrac{12y - 4x}{12x - 30y} \times \dfrac{-4x^2 + 14xy - 10y^2}{8x - 24y}$

$= \dfrac{-4\cancel{(x-3y)}}{6(2x-5y)} \times \dfrac{-2(2x^2 - 7xy + 5y^2)}{8\cancel{(x-3y)}}$

$= \dfrac{8\cancel{(2x-5y)}(x-y)}{48\cancel{(2x-5y)}}$

$= \dfrac{(x-y)}{6}$

3. $\dfrac{a^2 - ab - 2b^2}{a^2 + 2ab + b^2} \div \dfrac{a^2 - 4ab + 4b^2}{a+b}$

$= \dfrac{(a-2b)(a+b)}{(a+b)^2} \div \dfrac{(a-2b)^2}{(a+b)}$

$= \dfrac{\cancel{(a-2b)}\cancel{(a+b)}}{\cancel{(a+b)}\cancel{(a+b)}} \times \dfrac{\cancel{(a+b)}}{\cancel{(a-2b)}(a-2b)}$

$= \dfrac{1}{a-2b}$

4. $\dfrac{5b+5}{2b^2 - b - 3} \times \dfrac{6-4b}{5b^2 + 10b + 5} \div \dfrac{2a + 4ab}{2b^2 + 3b + 1}$

$= \dfrac{5(b+1)}{(2b-3)(b+1)} \times \dfrac{-2(2b-3)}{5(b+1)^2} \div \dfrac{2a(1+2b)}{(2b+1)(b+1)}$

$= \dfrac{5\cancel{(b+1)}}{\cancel{(2b-3)}\cancel{(b+1)}} \times \dfrac{-2\cancel{(2b-3)}}{5\cancel{(b+1)}(b+1)} \times \dfrac{\cancel{(2b+1)}\cancel{(b+1)}}{2a\cancel{(2b+1)}}$

$= \dfrac{-10}{10a(b+1)}$

$= \dfrac{-1}{a(b+1)}$

ALGEBRAIC FRACTIONS

2. Simplification of fractions with addition and Subtraction.

STEPS:

1. Factorise the denominator(s) (and numerator(s) where necessary)
2. Cancel like factors in each term if any.
3. Find the Lowest Common denominator (LCD)
4. Put each term over LCD by creating equivalent fractions.
5. Carry out the products in the numerator and add like terms.
6. Factorise numerator if possible and cancel any like factors.

1. $\dfrac{3x^2}{x^2 - x - 6} - \dfrac{3}{x-3} - \dfrac{3x}{x+2}$

$= \dfrac{3x^2}{(x-3)(x+2)} - \dfrac{3}{(x-3)} - \dfrac{3x}{(x+2)}$

$= \dfrac{3x^2 - 3x - 6 - 3x^2 + 9x}{(x-3)(x+2)}$

$= \dfrac{6x - 6}{(x-3)(x+2)}$

$= \dfrac{6(x-1)}{(x-3)(x+2)}$

2. $\dfrac{3a-4}{3a^2 - a - 4} - \dfrac{4a}{a^2 - 2a - 3}$

$= \dfrac{\cancel{(3a-4)}}{\cancel{(3a-4)}(a+1)} - \dfrac{4a}{(a-3)(a+1)}$

$= \dfrac{1(a-3) - 4a}{(a+1)(a-3)}$

$= \dfrac{-3a - 3}{(a+1)(a-3)}$

$= \dfrac{-3\cancel{(a+1)}}{\cancel{(a+1)}(a-3)}$

$= \dfrac{-3}{(a-3)}$

3. Restrictions on Fractions

All fractions with variables in their denominators will have restrictions, as a denominator may not equal zero. If the denominator becomes zero the fraction is undefined.

EXAMPLE:

Determine the value of x for which the fractions will be undefined:

1. $\dfrac{7x}{x-1}$
$x - 1 = 0$
$x = 1$

2. $\dfrac{3x-1}{2x+1}$
$2x + 1 = 0$
$x = \dfrac{1}{2}$

3. $\dfrac{2x-b}{3x-2b}$
$3x - 2b = 0$
$x = \dfrac{2b}{3}$

EXAMPLE:

Determine the restrictions on the following fractions:

1. $\dfrac{4x-2}{x^2-1}$
$x^2 - 1 \neq 0$
$x^2 \neq 1$
$x \neq \pm 1$

2. $\dfrac{3}{x-1} - \dfrac{4x}{(x+2)(x-1)}$
$x - 1 \neq 0$ and $x - 2 \neq 0$
$x \neq 1$ $\quad$ $x \neq -2$

EQUATIONS AND INEQUALITIES

SIMULTANEOUS EQUATIONS

Determine values that will satisfy both equations simultaneously. Two methods can be used: Elimination or Substitution.

EXAMPLE 1

$x - y = 8$ and $2x + y = 10$

ELIMINATION:

$x - y = 8$

$\underline{2x + y = 10}$

Add the equations to eliminate one of the variables

$3x = 18$

$x = \frac{18}{3}$

$x = 6$

$x - y = 8$

$(6) - y = 8$

$-y = 8 - 6$

$-y = -2$

$y = 2$

Substitute the value into the original equations to find the y-value

$\therefore (6; -2)$

SUBSTITUTION:

$x - y = 8 \ldots A$

$2x + y = 10 \ldots B$

$x = y + 8 \ldots C$

$2(y + 8) + y = 10$

$2y + 16 + y = 10$

$3y = 10 - 16$

$3y = -6$

$y = -2$

$x - y = 8$

$x - (-2) = 8$

$x + 2 = 8$

$x = 6$

Substitute the y-value into equation A to find the x-value

$\therefore (6; -2)$

The results of simultaneous equations are the point of intersection if the equations were to be presented graphically $(x; y)$.

EXAMPLE 2

$3x - 2y = 8$ and $4x + 2y = 6$

$3x - 2y = 8 \ldots A$

$4x + 2y = 6 \ldots B$

$4x = 2y = 6$

$2y = -4x + 6$

$\therefore y = -2x - 3 \ldots C$

$3x - 2y = 8$

$\therefore 3x - 2(-2x + 3) = 8$

$\therefore 3x + 4x - 6 = 8$

$\therefore 7x = 14$

$\therefore x = 2$

$y = -2x + 3$

$y = -2(2) + 3$

$y = -4 + 3$

$y = -1$

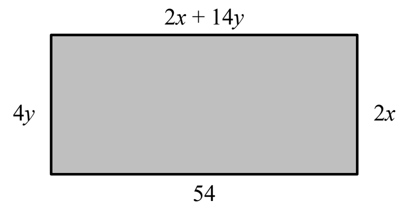
Substitute the x-value into equation C to find the y-value

$\therefore (2; -1)$

SIMULTANEOUS EQUATIONS: WORD PROBLEMS

EXAMPLE 1

Determine the values of x and y if the quadrilateral is a rectangle.



$4y = 2x \ldots A$

$2x + 14y = 54 \ldots B$

$y = \frac{1}{2}x \ldots C$

$2x + 14\left(\frac{1}{2}x\right) = 54$

$2x + 7x = 54$

$9x = 54$

$x = 6$

Substitute the y-value into A

$4y = 2(6)$

$y = \frac{12}{4}$

$y = 3$

EXAMPLE 2

The sweet you like is reduced by R2 on a special offer. This means you can get 14 sweets for the same price as you used to pay or 10. What is the usual price?

Usual price: x

Special price: $(x - 2)$

$10x = 14(x - 2)$

$10x = 14x - 28$

$28 = 14x - 10x$

$28 = 4x$

$7 = x$

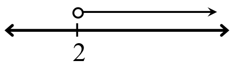
$\therefore$ the usual cost of the sweet is R7.

LINEAR INEQUALITIES:

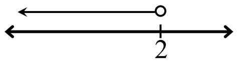
Relationship between expressions that are not equal

Inequality	**Interval Notation for Open Intervals**
$x > a$: x is greater than a	(a, ∞)
$x < a$: x is less than a	$(-\infty, a)$

$x > 2$

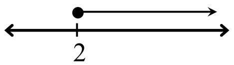
(2;∞) 

$x < 2$

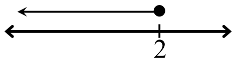
(−∞;2) 

Inequality	**Interval Notation for Closed Intervals**
$x \geq a$: x is greater than or equal to a	$[a, \infty)$
$x \leq a$: x is less than or equal to a	$(-\infty, a]$

$x \geq 2$

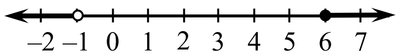
[2;∞) 

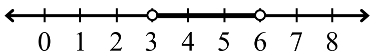
$x \leq 2$

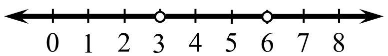
(−∞;2] 

EXAMPLE 1

Write down the inequality for each of the following:

a. 

b. 

c. 

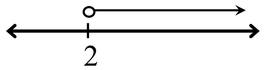
a. $x < -1$ or $x \geq 6$
$x \in (-\infty; -1)$ of $x \in [6; \infty)$

b. $3 < x < 6$
$x \in (3; 6)$

c. $-\infty < x < \infty,\ x \neq 3,\ x \neq 6$
$x \in (-\infty; \infty);\ \ x \neq 3,\ x \neq 6$

EXAMPLE 2

$4(2x - 1) > 5x + 2$
$8x - 4 > 5x + 2$
$8x - 5x > 2 + 4$
$3x > 6$
$\therefore x > 2$



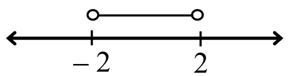
EXAMPLE 3

$-2 < 3 - \frac{1}{2}x \leq 5$ subtract 3 from all terms
$-5 < -\frac{1}{2}x \leq 2$ multiply all terms by -2
$10 > x \geq -4$
$\therefore -4 \leq x < 10$

NOTE: the inequality signs had to be REVERSED

EXAMPLE 4

$-3 < 2x + 1 < 5;\ x \in R$ subtract 1 from all terms
$-3 - 1 < 2x + 1 - 1 < 5 - 1$
$-4 < 2x < 4$ divide all terms by 2
$\therefore -2 < x < 2$



EQUATIONS AND INEQUALITIES

REMINDERS:

1. Linear equation: an equation of degree one with at most one solution
2. Distributive law: $a(b+c) = ab + ac$; that is, the monomial factor a is distributed, or separately applied, to each term of the binomial factor $b + c$
3. Like terms: terms that have the same variables and powers
4. Quadratic Equation: an equation of the second degree
5. Degree of the equation: the exponent of the highest power to which that variable is raised in the equation
6. Constant term: a known, fixed value
7. Coefficient: a number used to multiply a variable
8. Inequality: <,> two values that are not equal.
9. Fractions and 0:

$$\frac{\text{Numerator}}{\text{Denominator}}$$

$$\frac{0}{x} = 0 \text{ BUT } \frac{x}{0} = \text{undefined}$$

10. Multiplication of signs:

$(-) \times (+) = (-)$
$(-) \times (-) = (+)$

LINEAR EQUATIONS (Gr9 Revision)

Move all the variables to the one side ad the constants to the other on order to solve. Linear equations have only one solution.

EXAMPLE 1
Solve for p:

$4(2p-7) - 8(5-p) = 3(2p+4) - 5(p+7)$	Use the distributive law
$8p - 28 - 40 + 8p = 6p + 12 - 5p - 35$	Group like terms
$8p + 8p - 6p + 5p = 28 + 40 + 12 - 35$	Simplify
$15p = 45$	
$p = 3$	

EXAMPLE 2
Solve for a:

$\frac{2a-1}{5} - a + 5 = 0$	Find LCD (5)
$\left(\frac{2a-1}{5}\right) \times 5 - (a) \times 5 + (5) \times 5 = 0 \times 5$	Multiply both sides by LCD
$2a - 1 - 5a + 25 = 0$	Group like terms
$2a - 5a = -25 + 1$	Simplify
$-3a = -24$	
$a = \frac{-24}{-3}$	
$a = 8$	

LITERAL EQUATIONS

Make a specific variable the subject of the equation

EXAMPLE

The surface area of a cylinder is given by $A = 2\pi r(h + r)$.

Prove $h = \frac{A - 2\pi r}{2\pi r}$

$A = 2\pi r(h+r)$	Manipulate the equation for find h on its own
$\frac{A}{2\pi r} = h + r$	
$\frac{A}{2\pi r} - r = h$	
$\frac{A}{2\pi r} - \frac{r \times 2\pi r}{2\pi r} = h$	Adding fractions (LCD is $2\pi r$)
$\frac{A - 2\pi r^2}{2\pi r} = h$	

EQUATIONS AND INEQUALITIES

FRACTIONS WITH VARIABLES IN THE DENOMINATOR

Steps for solving unknowns in the denominator:

1. Factorise denominators, apply the sign-change rule if necessary.
2. State restrictions.
3. Multiply every term by the lowest common denominator (LCD).
4. Solve the equation.

EXAMPLE 1

$\frac{3}{x-2} + \frac{x+3}{4-x^2} = \frac{6}{x+2}$ Change sign to simplify factorisation

$\frac{3}{x-2} - \frac{x+3}{x^2-4} = \frac{6}{x+2}$ Factorise denominators

$\frac{3}{x-2} - \frac{x+3}{(x-2)(x+2)} = \frac{6}{x+2}$ LCD: $(x-2)(x+2); x \neq 2, x \neq -2.$ Multiply every term by the LCD

$\frac{3}{x-2} \times (x-2)(x+2) - \frac{x+3}{(x-2)(x+2)} \times (x-2)(x+2) = \frac{6}{x+2} \times (x-2)(x+2)$

$3 \times (x+2) - (x+3) = 6 \times (x-2)$

$3x + 6 - x - 3 = 6x - 12$ Simplify

$6 - 3 + 12 = 6x - 3x + x$

$15 = 4x$

$x = \frac{15}{4}$

EXAMPLE 2

$\frac{2x}{x-3} + \frac{5x-3}{9-x^2} = \frac{x}{x+3}$ Change sign to simplify factorisation

$\frac{2x}{x-3} - \frac{5x-3}{x^2-9} = \frac{x}{x+3}$ Factorise denominators

$\frac{2x}{x-3} - \frac{5x-3}{(x-3)(x+3)} = \frac{x}{x+3}$ LCD: $(x-3)(x+3); x \neq 3, x \neq -3.$ Multiply every term by the LCD

$\frac{2x}{x-3} \times (x-3)(x+3) - \frac{5x-3}{(x-3)(x+3)} \times (x-3)(x+3) = \frac{x}{x+3} \times (x-3)(x+3)$

$2x(x+3) - (5x-3) = x(x-3)$

$2x^2 + 6x - 5x + 3 = x^2 - 3x$

$x^2 + 4x + 3 = 0$ Simplify

$(x+1)(x+3) = 0$

$(x+1) = 0$ or $(x+3) = 0$

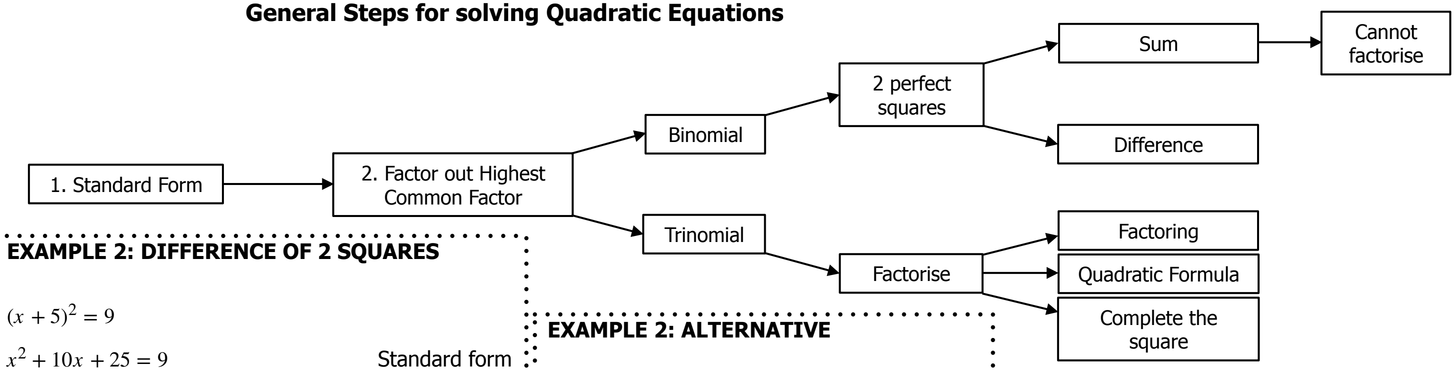
$x = -1$ or $x = -3$

BUT $x \neq -3 \therefore x = -1$ is the only solution

QUADRATIC EQUATIONS

$ax^2 + bx + c = 0 \quad where\ a \neq 0$

General Steps for solving Quadratic Equations



EXAMPLE 1: FACTORISATION

$5x + 2x^2 = 3$ Standard form

$2x^2 + 5x - 3 = 0$ Factorise

$(2x-1)(x+3) = 0$ Apply the zero-factor law*

$2x - 1 = 0$ or $x + 3 = 0$

$2x = 1$ or $x = -3$

$x = \frac{1}{2}$

$\therefore x = \frac{1}{2}$ or $x = -3$

EXAMPLE 2: DIFFERENCE OF 2 SQUARES

$(x+5)^2 = 9$

$x^2 + 10x + 25 = 9$ Standard form

$x^2 + 10x + 16 = 0$ Factorise

$(x+8)(x+2) = 0$ Apply zero-factor law*

$x + 8 = 0$ or $x + 2 = 0$

$x = -8$ or $x = -2$

EXAMPLE 2: ALTERNATIVE

$(x+5)^2 = 9$

$\sqrt{(x+5)^2} = \sqrt{9}$

$x + 5 = \pm 3$

$x + 5 - 3 = 0$ or $x + 5 + 3 = 0$

$x + 2 = 0$ or $x + 8 = 0$

$x = -2$ or $x = -8$

***Zero-Factor Law:**
if $a \times b = 0$ then $a = 0$ or $b = 0$

EXPONENTS

LAWS OF EXPONENTS

Laws of exponents only apply to multiplication, division, brackets and roots. NEVER adding or subtracting. For the following: $a, b > 0$ and $m, n \in Z$

1. $a^m \times a^n = a^{m+n}$
2. $\dfrac{a^m}{a^n} = a^{m-n}$
3. $(a^m \cdot a^n)^p = a^{n\times p} \cdot a^{m\times p}$
4. $\sqrt[n]{a^m} = a^{\frac{m}{n}}$

NOTES:

1. $a^{-n} = \dfrac{1}{a^n}$ **or** $a \cdot b^{-m} = \dfrac{a}{b^m}$

2. $a^0 = 1$

MULTIPLICATION AND DIVISION

EXAMPLES

Simplify, leaving all answers in positive exponential form.

1. [Multiplication bases variables]

$$\frac{(a^4b^2)^2 \cdot a^4b^3}{(ab^3)^4 \cdot \sqrt[3]{a^{-6}}} \qquad \text{Laws 3 \& 4}$$

$$= \frac{a^8b^4 \cdot a^4b^3}{a^4b^{12} \cdot a^{-2}} \qquad \text{Law 1}$$

$$= \frac{a^{12}b^7}{a^2b^{12}} \qquad \text{Law 2}$$

$$= a^{10}b^{-5} \qquad \text{Note 1}$$

$$= \frac{a^{10}}{b^5}$$

2. [Multiplication bases numerical]

$$\frac{18^{2n+1} \cdot 9^{\frac{1}{2}} \cdot 16^{n-1}}{81^{n-3} \cdot 64^{n+2}} \qquad \text{Change to prime bases}$$

$$= \frac{(3^2 \cdot 2)^{2n+1} \cdot (3^2)^{\frac{1}{2}} \cdot (2^4)^{n-1}}{(3^4)^{n-3} \cdot (2^6)^{n+2}} \qquad \text{Law 3}$$

$$= \frac{3^{4n+2} \cdot 2^{2n+1} \cdot 3 \cdot 2^{4n-4}}{3^{4n-12} \cdot 2^{6n+12}} \qquad \text{Law 1}$$

$$= \frac{2^{6n-3} \cdot 3^{4n+3}}{2^{6n+12} \cdot 3^{4n-12}} \qquad \text{Law 2}$$

$$= 2^{6n-3-(6n+12)} \cdot 3^{4n+3-(4n-12)}$$

$$= 2^{-15} \cdot 3^{15} \qquad \text{Note 1}$$

$$= \frac{3^{15}}{2^{15}}$$

$$= \left(\frac{3}{2}\right)^{15}$$

3. [Multiplication bases combined]

$$\frac{(27m^{-1})^{\frac{1}{3}} \times (m^{\frac{1}{3}})^4}{(81m^2)^{-\frac{1}{2}}} \qquad \text{Change to prime bases}$$

$$= \frac{(3^3m^{-1})^{\frac{1}{3}} \times (m^{\frac{1}{3}})^4}{(3^4m^2)^{-\frac{1}{2}}} \qquad \text{Law 3}$$

$$= \frac{3m^{-\frac{1}{3}} \times m^{\frac{4}{3}}}{3^{-2}m^{-1}} \qquad \text{Laws 1 \& 2}$$

$$= 3^{1+2}m^{-\frac{1}{3}+\frac{4}{3}+1}$$

$$= 3^3m^2$$

$$= 27m^2$$

ADDITION AND SUBTRACTION

EXAMPLES

Simplify, leaving all answers in positive exponential form.

NOTE:
To be able to simplify algebraic expressions of 2 or more terms, one must always **factorise FIRST**

1.

$$\frac{2^{n-1} + 2^{n-2}}{2^n - 2^{n+2}} \qquad \text{Expand}$$

$$= \frac{2^n \cdot 2^{-1} + 2^n \cdot 2^{-2}}{2^n - 2^n \cdot 2^2} \qquad \text{Factorise using HCF and Note 1}$$

$$= \frac{\cancel{2^n}\left(\frac{1}{2} + \frac{1}{2^2}\right)}{\cancel{2^n}(1 - 2^2)} \qquad \text{Simplify}$$

$$= \left(\frac{2+1}{4}\right) \div (1-4) \qquad \text{Note 1}$$

$$= \frac{3}{4} \times \frac{1}{-3}$$

$$= -\frac{1}{4}$$

2.

$$\frac{5 \cdot 2^{2x-1} + 3 \cdot 4^{x+1}}{2^{2x} - 6 \cdot 2^{2x-2}} \qquad \text{Change to prime bases}$$

$$= \frac{5 \cdot 2^{2x} \cdot 2^{-1} + 3 \cdot 2^{2x} \cdot 2^2}{2^{2x} - 6 \cdot 2^{2x} \cdot 2^{-2}} \qquad \text{Change 4 to prime and expand}$$

$$= \frac{\cancel{2^{2x}}\left(5 \cdot \frac{1}{2} + 3 \cdot 2^2\right)}{\cancel{2^{2x}}\left(1 - 6 \cdot \frac{1}{4}\right)} \qquad \text{Take out common factor}$$

$$= \frac{\frac{5}{2} + 12}{1 - \frac{3}{2}} \qquad \text{Simplify bracketed terms}$$

$$= \frac{\frac{29}{2}}{-\frac{1}{2}}$$

$$= \frac{29}{2} \times -\frac{2}{1} \qquad \text{Tip and times}$$

$$= -29$$

ADDITION AND SUBTRACTION

EXAMPLES continued

Simplify, leaving all answers in positive exponential form.

> **NOTE:**
> To be able to simplify algebraic expressions of 2 or more terms, one must always **factorise FIRST**

3. $\frac{4^x - 4}{2^x - 2}$ — Change 4^x to prime

$= \frac{2^{2x} - 4}{2^x - 2}$ — Numerator is Diff of 2 Squares (DOTS)

$= \frac{(2^x - 2)(2^x + 2)}{(2^x - 2)}$

$= 2^x + 2$

4. $\frac{2^{1026} - 2^{1024}}{\sqrt{2^{2044}}}$ — HCF and Law 4

$= \frac{2^{1024}(2^2 - 1)}{2^{1022}}$ — Law 2 and simplify bracket

$= 2^{1024-1022}(4 - 1)$

$= 2^2(3)$

$= 12$

EXPONENTIAL EQUATIONS

EXAMPLES

Solve for x

1. $3x^{\frac{2}{3}} = 12$ — Divide by 3

$x^{\frac{2}{3}} = 4$ — Raise both sides to the inverse power

$(x^{\frac{2}{3}})^{\frac{3}{2}} = (2^2)^{\frac{3}{2}}$

$x = \pm 8$

2. $2 \cdot 3^{x-1} \cdot 3^{2x+2} - 4 = 50$ — Isolate powers

$2 \cdot 3^{3x+1} = 50 + 4$ — Simplify LHS

$3^{3x+1} = 54 \div 2$ — Divide both sides by 2 (as it has no exponent)

$3^{3x+1} = 27$ — Get bases the same by using prime factors

$3^{3x+1} = 3^3$ — If bases same, exponents must be same to be =

$\therefore 3x + 1 = 3$

$x = \frac{2}{3}$

Equations involving factorisation

3. $3 \cdot 5^{2x-1} - 3 = 0$

$3 \cdot 5^{2x-1} = 3$

$5^{2x-1} = 1$

$\therefore 5^{2x-1} = 5^0$ — Remember that $5^0 = 1$ (Note 2)

$\therefore 2x - 1 = 0$

$x = \frac{1}{2}$

4. $2^{x+3} = 2^x + 28$

$2^{x+3} - 2^x = 28$ — Get powers together

$2^x \cdot 2^3 - 2^x = 28$

$2^x(2^3 - 1) = 28$ — Factorise LHS

$2^x(7) = 28$ — Simplify bracket & divide

$2^x = 4$ — Change 4 to prime

$2^x = 2^2$

$\therefore x = 2$

NUMBER PATTERNS

TERMINOLOGY:

1. Consecutive: numbers or terms following directly after each other
2. Common/constant difference: the difference tween two consecutive terms.
$$d = T_2 - T_1$$
$$d = T_3 - T_2$$
3. Terms are indicated by a T and the position or number of the term in the pattern by a sub-script, e.g. term 1 is T_1 or term 50 is T_{50} .
4. General term T_n: also referred to as the n^{th} term.
   - General term for linear patterns:
$$T_n = dn + c$$

Linear Patterns:

Sequences with a constant difference between the terms.

$$T_n = dn + c$$

T_n = general term
d = constant difference
n = number of the term

Steps to determine the n^{th} term:

1. Find the constant difference
$d = T_2 - T_1 = T_3 - T_2$
2. Find the c-value
$c = T_1 - d$
3. Substitute the c- and d-values to define the n^{th} term.

Quadractic Patterns:

By inspection:

$$T_n = an^2 + c$$

T_n = general term
a = constant difference ÷ 2
n = number of the term

EXAMPLE

i. Given the sequence 3, 8, 13, …
ii. Determine the next three terms.
iii. Determine the general term.
iv. Determine the value of the 45^{th} term.
v. Which term has the value of 403?

SOLUTION

i. 18; 23; 28

ii. $d = 8 - 3 = 5$ or $d = 13 - 8 = 5$

$c = T_1 - d$
$= 3 - 5$
$= -2$

$\therefore T_n = dn + c$
$T_n = 5n - 2$

iii. $n = 45$
$\therefore T_{45} = 5(45) - 2$
$= 223$

iv. $T_n = 403$ so need to solve for n

$T_n = 5n - 2$
$5n - 2 = 403$
$5n = 405$
$\therefore n = 81$

EXAMPLE

Mpho is told that a sequence has a n^{th} term of $15n - 2$. She has to find which term will be equal to 96. She is stuck because she keeps getting an unexpected answer. Perform the calculations and then explain the answer

SOLUTION

$T_n = 96$
$\therefore 15n - 2 = 96$
$15n = 98$
$n = 6\frac{8}{15}$

$\therefore$ 96 is not a term in the sequence since $n \in \mathbb{N}$

EXAMPLE

Given $T_6 = 8$ and $T_9 = -1$, determine T_5 and T_7 .

SOLUTION

T_5; 8; T_7; T_8; -1

$T_7 = T_6 + d$
$= 8 + d$

$T_8 = T_7 + d$
$= (8 + d) + d$
$= 8 + 2d$

$T_9 = T_8 + d$
$-1 = (8 + 2d) + d$
$-9 = 3d$
$-3 = d$

$\therefore T_5 = T_6 - d$
$= 8 - 3$
$= 11$

and

$T_7 = T_6 + d$
$= 8 + (-3)$
$= 5$

EXAMPLE

If the pattern "safesafesafesafe…" continues in this way what would the 263rd letter be?

SOLUTION

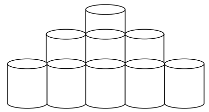
Note safe has 4 letters so safesafe has 8 and safesafesafe has 12 etc.

$\therefore 263 \div 4 = 65$ remainder 3

Thus 65 safe and three more letters, the 263rd letter is f.

EXAMPLE

A shop owner wishes to display cans of food in a triangular shape as shown in figure. There is one can in the top row, three in the second row and so on.



i. Write down the first four terms of this pattern.
ii. In what type of sequence are the tins arranged?
iii. Write down a formula for the term of the sequence.
iv. How many cans are needed for the 15th row?
v. In which row will there be 27 cans?

SOLUTION

i. 1; 3; 5; 7

ii. Linear sequence

iii. $d = 3 - 1 = 2$ or $d = 5 - 3 = 2$

$c = T_1 - d$
$= 1 - 2$
$= -1$

$\therefore T_n = dn + c$
$T_n = 2n - 1$

iv. $t_{15} = 2(15) - 1$
$= 29$
$\therefore$29 cans are needed for the 15^{th} row

v. $T_n = 27$
$2n - 1 = 27$
$\therefore n = 14$ ($\therefore 14^{th}$ row)

FINANCE - SIMPLE AND COMPOUND INTEREST

SIMPLE INTEREST

Interest calculated on only the money initially invested or borrowed.

$$A = P(1 + in)$$

COMPOUND INTEREST

Interest calculated on the initial amount and any subsequent interest that is earned or charged.

$$A = P(1 + i)^n$$

A = accumulated amount	P = original amount
n = number of years	i = interest rate $\frac{r}{100}$

In financial Maths, unless instructed otherwise, always round off your **FINAL answer** to **2 decimal places**

EXAMPLE 1

Which investment would be a better option over 6 years?

a) 6% p.a. simple interest

$A = P(1 + in)$

$= P(1 + 0{,}06 \times 6)$

$= 1{,}36P$

b) 5,5% p.a. compound interest

$A = P(1 + i)^n$

$= P(1 + 0{,}055)^6$

$= 1{,}38P$

$\therefore$ Option (b) is better

EXAMPLE 2

John wants to have R10 000 available in 4 years' time for a holiday. How much does he need to invest now if he can get an interest rate of 8,3% p.a. compounded annually?

$A = P(1 + i)^n$

$10\ 000 = P(1 + 0{,}083)^4$

$10\ 000 = P(1{,}375...)$ **DO NOT round off yet!**

$P = \text{R}7\ 269{,}20$

EXAMPLE 3

You want to double an investment of R1 200 in five years. What annual interest would yield this return?

a) Simple interest

$A = P(1 + in)$

$2400 = 1200(1 + i \times 5)$

$2 = 1 + 5i$

$1 = 5i$

$\therefore i = 0{,}2$

$= 20{,}00$ % p.a.

a) Compound interest

$A = P(1 + i)^n$

$2400 = 1200(1 + i)^5$

$2 = (1 + i)^5$

$\sqrt[5]{2} = 1 + i$

$1 + i = 1{,}14869...$

$\therefore i = 0{,}14869...$

$= 14{,}87$% p.a.

HIRE PURCHASE LOANS

Short-term loans to buy goods on credit, normally repaid in equal monthly installments.

$$A = P(1 + in)$$

EXAMPLE

Tania wants to buy a new TV which costs R9 350. She can't afford the full amount now and agrees to buy it on the following hire purchase terms:

➡ 12% deposit ➡ 13% interest p.a. ➡ equal monthly installments over 2 years

a) How much is the deposit amount?

$12\,\%$ of R9 350 = R1 122

b) How much will Tania repay, including interest?

$\text{R}9\ 350 - \text{R}1\ 122 = \text{R}8\ 228$

$A = P(1 + in)$

$= 8\ 228(1 + 0{,}13 \times 2)$

$= \text{R}10\ 367{,}28$

c) How much will her monthly repayments be?

$\text{R}10\ 367{,}28 \div 24 = \text{R}431{,}97$ per month

d) How much would Tania have saved if she could have paid the full amount initially?

$(\text{R}1\ 122 + \text{R}10\ 367{,}28) - \text{R}9\ 350 = \text{R}2\ 409{,}28$

INFLATION

The rising cost of goods and services

$$A = P(1 + i)^n$$

EXAMPLE

A loaf of bread costs R14. If the average inflation rate has been 8%, and assuming it remains constant:

a) How much will a loaf of bread cost in 5 years?

$A = P(1 + i)^n$

$= 14(1 + 0{,}08)^5$

$= \text{R}20{,}57$

b) How much did a loaf of bread cost 13 years ago?

$A = P(1 + i)^n$

$14 = P(1 + 0{,}08)^{13}$

$P = \text{R}5{,}15$

POPULATION GROWTH

$$A = P(1 + i)^n$$

EXAMPLE

A herd of cows was made up of 24 animals in 2014. If the growth of the herd was approximately 13% p.a., how many cows would you have expected in the herd in 2018?

$A = P(1 + i)^n$

$= 24(1 + 0{,}013)^4$

$= 39{,}13$

≈ 39 cows

FINANCE

FOREIGN EXCHANGE RATES

CURRENCY	RATE OF EXCHANGE (OF THE RAND)
Pound (£)	18,23
US Dollar ($)	15,42

EXAMPLE

a) A South African lady working in London manages to save R4 000 per month. How many pounds does she save in a year?

$R4\ 000 \div R18{,}23/£ \times 12$

$= £2\ 633{,}02$

b) If she wanted to buy a book from America, on Amazon, for $15, how much would she pay in pounds?

$\$15 \times R15{,}42/\$$

$= R231{,}30$

$R231{,}30 \div R18{,}23/£$

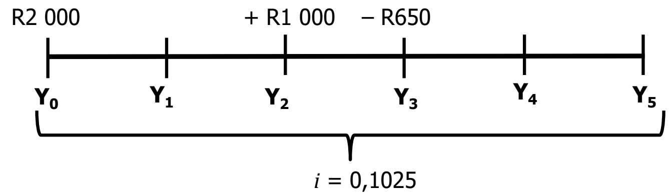
$= £12{,}69$

TIMELINES

Timelines can be used to visually represent more complicated situations in Financial Maths

TYPE 1: MONEY IN AND OUT

Thando invests R2 000 in a fixed deposit. Two years later, he adds R1 000. One year after that, he needs to withdraw R650.
If the interest rate is 10,25% p.a. compounded annually for the entire period, how much money will Thando have after 5 years?



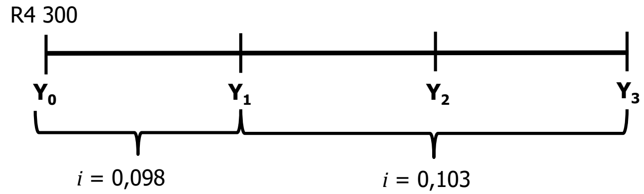
$A = P(1+i)^n$

$= 2000(1+0{,}1025)^5 + 1000(1+0{,}1025)^3 - 650(1+0{,}1025)^2$

$= R3\ 807{,}81$

TYPE 2: CHANGE OF INTEREST RATE

Sam deposits R4 300 into a 3 year fixed deposit account. The interest (all compounded annually) is 9,87% p.a. for the first year, and then 10,3% p.a. for the remainder of the period. How much will Sam have at the end of 3 years?



$A = P(1+i)^n$

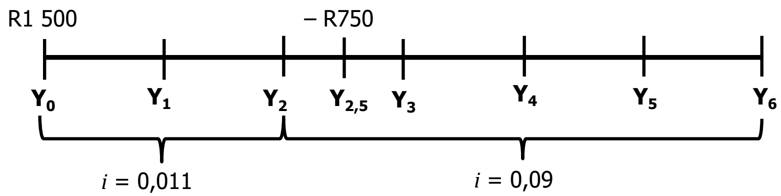
$= 4300(1+0{,}098)^1(1+0{,}103)^2$

$= R5\ 744{,}10$

MULTIPLY when the rate changes

TYPE 3: COMBINATIONS

Jenny invests R1 500, but two and a half years later, she needs to withdraw half of the initial investment. The interest rate for the first two years is 11% p.a. compound interest and 9% p.a. compound interest for the other 4 years. How much money will Jenny have after 6 years?



$A = P(1+i)^n$

$= 1500(1+0{,}011)^2(1+0{,}09)^4 - 750(1+0{,}09)^{3,5}$

$= R1\ 594{,}78$

FUNCTIONS AND GRAPHS

FUNCTION

A **FUNCTION** is a rule by means of which each element of the domain (independent variable or input value(s), i.e. x) is associated with **one and only one** element of the range (dependent variable or output value(s), i.e. y)

Functions can be represented in different ways for example:

$y = 2x - 3$; $y = 2x^2 + 1$; $xy = 4$ or 2

or

$f(x) = -5x + 1$; $g(x) = x^2 - 5$

Example 2 is known as function notation and is an easier way of representing the y-value. $\therefore y = f(x)$

So we can write $y = 3x + 1$ as $f(x) = 3x + 1$ and is read as follows; the value of the function f at x is equal to $3x + 1$, where $f(x)$ is the range and x is the domain. Thus $f(2)$ will give the output value when 2 is substituted in for x, i.e. $f(2) = 3(2) + 1 = 7$ so ordered pair (2; 7).

EXAMPLE:

If $g(x) = 3x^2 - 5x$, determine;

$g(-1)$

$g(x) = 2$

$g(3) - g(-2)$

SOLUTION:

x is given, solve for y

$g(-1)$

$= 3(-1)^2 - 5(-1)$

$= 8$

y given so solve for x

$g(x) = 2$

$3x^2 - 5x = 2$

$3x^2 - 5x - 2 = 0$

$(3x + 1)(x - 2) = 0 \quad \therefore x = -\frac{1}{3} \quad$ or $\quad x = 2$

$g(3) - g(-2)$

$= (3(3)^2 - 5(3)) - (3((-2)^2 - 5(-2))$

$= 12 - 22$

$= -10$

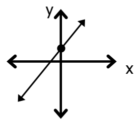
TYPES OF FUNCTIONS AND THEIR GRAPHS - LINEAR FUNCTION

1. LINEAR FUNCTION (Straigh line graph)

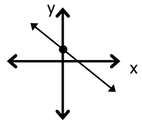
$y = mx + c$ or $y = ax + q$

- m = a = gradient or slope

$y = mx + c$ or $m > 0$



$y = -mx + c$ or $m < 0$



$m = \frac{y_2 - y_1}{x_2 - x_1}$

c = q = y-intercept i.e. (0;c)

- Domain: $x \in \mathbb{R}$ Range: $y \in \mathbb{R}$

FINDING THE EQUATION

Remember m can be found in the following ways:

1. given two coordinates: $m = \frac{y_2 - y_1}{x_2 - x_1}$
2. Function f parallel to function g: $m_f = m_g$
3. Function f perpendicular to function g: $m_f \times m_g = -1$

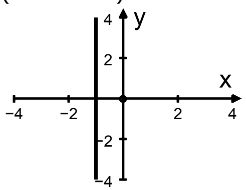
EXAMPLE:

Determine the equation of the straight line passing through (–1;4) and(–1;–2).

SOLUTION:

$m = \frac{4 - (-2)}{-1 - (-1)} = \text{undefined}$

$\therefore x = -1$ (vertical line)



EXAMPLE:

Determine the equation of the straight line passing through (–1;–1) and perpendicular to $x - 4y = 4$.

SOLUTION:

$x - 4y = 4$ put in standard form

$y = \frac{1}{4}x - 1$

$\therefore m_f = \frac{1}{4}$

Graphs are perpendicular,

$\therefore m_f \times m_g = -1$

$\therefore m_g = -4$

Thus $y = -4x + c$ sub into (–1;–1)

$-1 = -4(-1) + c$

$\therefore c = 3$

$\therefore y = -4x + 3$

EXAMPLE:

Determine the equation of the straight line passing through A(–4;3) and B(3;–2).

SOLUTION:

$m = \frac{y_2 - y_1}{x_2 - x_1}$

$m = \frac{-2 - 3}{3 - (-4)}$

$m = -\frac{5}{7}$

$\therefore = -\frac{5}{7}x + c$

$3 = -\frac{5}{7}(-4) + c$

$\therefore c = \frac{1}{7}$

$\therefore y = -\frac{5}{7}x + \frac{1}{7}$

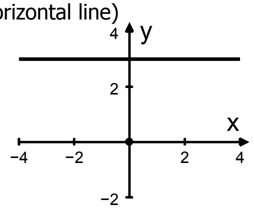
EXAMPLE:

Determine the equation of the straight line passing through (–1;3) and(–5;3).

SOLUTION:

$m = \frac{3 - 3}{-5 - (-2)} = 0$

$\therefore y = 3$ (horizontal line)



EXAMPLES:

Sketch the following graphs on same set of axes showing all intercepts with the axes:

a) $f(x) = -2x + 1$ using dual-intercept method.

y-intercept:

$(x = 0$ or $f(0)) : (0; 1)$

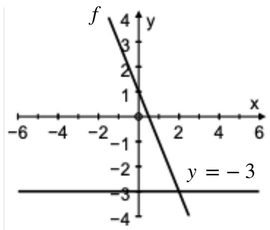
x-intercept:

$(y = 0$ or $f(x) = 0)$

$0 = -2x + 1$

$\therefore x = \frac{1}{2}$

b) $y = -3$



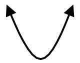
FUNCTIONS AND GRAPHS

TYPES OF FUNCTIONS AND THEIR GRAPHS - QUADRATIC FUNCTION (PARABOLA)

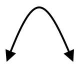
2. QUADRATIC FUNCTION (PARABOLA)

$y = ax^2 + q, \quad a \neq 0$

- $a > 0$ or a is +'ve



$a < 0$ or a is –'ve



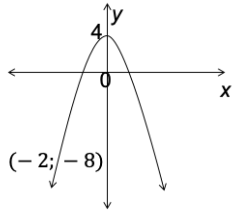
- q = y-intercept
- Turning point = (0;q)
- Domain: $x \in \mathbb{R}$
- Range: $y \in [p;\infty)$ minimum if $a > 0$

  $y \in (-\infty; p]$ maximum if $a < 0$
- Symmetry: $x = 0$

FINDING THE EQUATION

1. Given turning point (0, q) and another point use $y = ax^2 + q$.

EXAMPLE:
Determine equation of



SOLUTION:
First sub the turning points y-value in for q:

$y = ax^2 + 4$

now sub (–2;–8) to find a

$-8 = a(-2)^2 + 4$

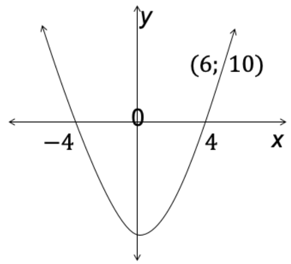
$-16 = 4a$

$\therefore a = -4$

$\therefore y = -4x^2 + 4$

2. Given roots (x-intercepts) and another point use $y = a(x - R_1)(x - R_2)$

EXAMPLE:
Determine equation of



SOLUTION:
First sub in the roots –4 and 4

$y = a(x - (-4))(x - 4)$ FOIL out

$y = a(x^2 - 16)$ now sub in (6;10)

$10 = a(6^2 - 16)$

$10 = 20a$

$\therefore a = \frac{1}{2}$

$\therefore y = \frac{1}{2}(x^2 - 16)$

$\therefore y = \frac{1}{2}x^2 - 8$

SKETCHING THE GRAPHS

Steps:

1. Find Turning point or y-intercept.
2. Find x-intercepts and in none use the table method.
3. Determine the shape.

EXAMPLE:
Sketch the following graph showing all intercepts with the axes and turning points: $f(x) = 2x^2 + 2$

SOLUTION:
Turning point or y-intercept:

(0;2)

x-intercepts (y=0):

$2x^2 + 2 = 0$

$2x^2 = -2$ has no solution

$\therefore$ has no x – intercepts

Shape:

a > 0;

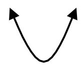
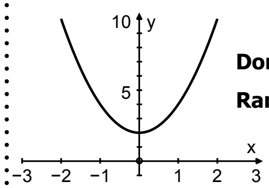


Table of data points (on calculator):

x	–2	–1	0	1	2
y	10	4	2	4	10



Domain: $x \in \mathbb{R}$

Range: $y \in [2;\infty)$

EXAMPLE:
Sketch the following graph showing all intercepts with the axes and turning points: $g(x) = -2x^2 + 8$

SOLUTION:
Turning point or y-intercept:

(0;8)

x-intercepts (y=0):

$0 = -2x^2 + 8$

$2x^2 = 8$

$x^2 = 4$

$x = \pm\sqrt{4}$

$x = 2 \quad or \quad x = -2$

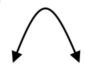
OR

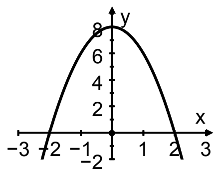
$2(x^2 - 4) = 0$

$(x - 2)(x + 2) = 0$

Shape:

a < 0;





Domain: $x \in \mathbb{R}$

Range: $y \in (\infty; 8]$

FUNCTIONS AND GRAPHS

TYPES OF FUNCTIONS AND THEIR GRAPHS - HYPERBOLA

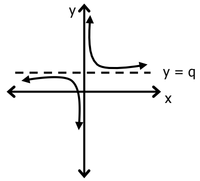
HYPERBOLA

$y = \frac{a}{x} + q, \quad a \neq 0$

- a = constant and affects the shape

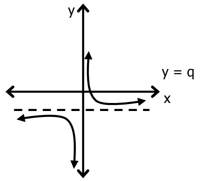
$y = \frac{+a}{x} + q$

$a > 0; \quad q > 0$



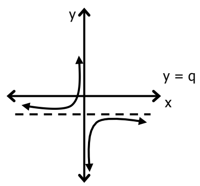
$y = \frac{+a}{x} - q$

$a > 0; \quad q < 0$



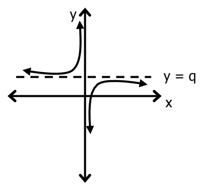
$y = \frac{-a}{x} - q$

$a < 0; \quad q < 0$



$y = \frac{-a}{x} + q$

$a < 0; \quad q > 0$



- q = constant but shifts graph up or down the y-axis.
- Asymptotes: $x = 0; \quad y = 0$
  (values that make the function undefined)
- Domain: $x \in \mathbb{R}, \quad x \neq 0$
- Range: $y \in \mathbb{R}, \quad y \neq q$
- Axis of symmetry: $y = x + q$ or $y = -x + q$

SKETCHING THE GRAPHS

Steps:

1. Determine the asymptotes ($x = 0$ and $y = q$)
2. Determine the x-intercepts
3. Determine the shape
4. Use table method to plot at least 3 other points

EXAMPLE:
Sketch the following graph showing all intercepts with the axes and asymptotes:
$f(x) = -\frac{3}{x}$

SOLUTION:
Asymptotes:

$x = 0; \quad y = 0$

As $y = 0$ is an asymptote there are no x-axis intercepts.

Shape:

$a < 0$

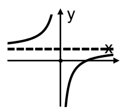
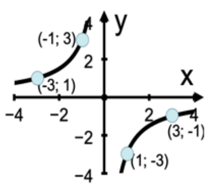


Table:

x	–3	–1	1	3
y	1	3	–3	–1



Domain: $x \in \mathbb{R}, x \neq 0$

Range: $y \in \mathbb{R}, y \neq 0$

Lines of symmetry: $y = x \quad or \quad y = -x$

EXAMPLE:
Sketch the following graph showing all intercepts with the axes and asymptotes:
$h(x) = \frac{2}{x} + 1$

SOLUTION:
Asymptotes:

$x = 0; \quad y = 1$

x-intercepts: $0 = \frac{2}{x} + 1$

$0 = 2 + x$

$x = -2$

Shape:

$a > 0$

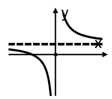
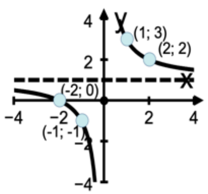


Table:

x	–2	–1	1	2
y	0	–1	3	2



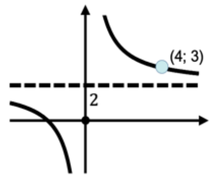
Domain: $x \in \mathbb{R}, x \neq 0$

Range: $y \in \mathbb{R}, y \neq 1$

Lines of symmetry: $y = x + 1 \; or \; y = -x + 1$

FINDING THE EQUATION

Determine the equation of:



$y = \frac{a}{x} + q$

First sub in asymptote $y = 2$ in place of q

$y = \frac{a}{x} + 2$

Sub in the coordinate $(4; 3)$

$3 = \frac{a}{4} + 2$

$12 = a + 8$

$\therefore a = 4$

$\therefore y = \frac{4}{x} + 2$

FUNCTIONS AND GRAPHS

TYPES OF FUNCTIONS AND THEIR GRAPHS - EXPONENTIAL FUNCTIONS

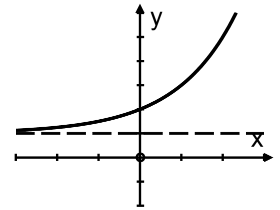
EXPONENTIAL FUNCTIONS

$y = a \cdot b^x + q \qquad a \neq 0$ and $b > 0,\ b \neq 1$

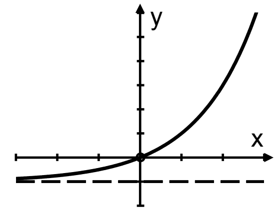
Asymptote: $y = q$ **Domain**: $x \in \mathbb{R}$

Range: $y \in (q;\infty)$ *if* $a > 0$; $y \in (-\infty;q)$ *if* $a < 0$

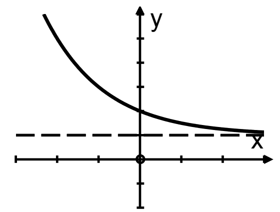
$y = +a \cdot b^x + q,\ b > 1$



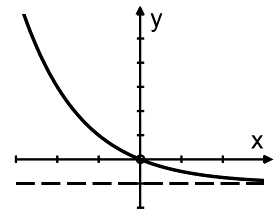
$y = +a \cdot b^x - q,\ b > 1$



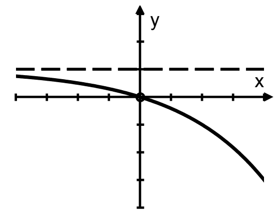
$y = +a \cdot b^x + q, 0 < b < 1$



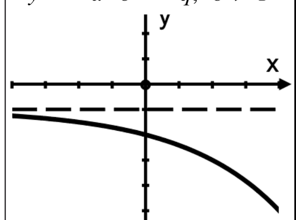
$y = +a \cdot b^x - q,\ 0 < b < 1$



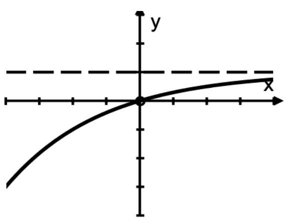
$y = -a \cdot b^{+x} + q,\ b > 1$



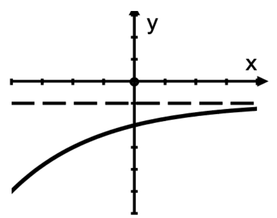
$y = -a \cdot b^{+x} - q,\ b > 1$



$y = -a \cdot b^x + q,\ 0 < b < 1$



$y = -a \cdot b^x - q,\ 0 < b < 1$



SKETCHING THE GRAPHS

Steps:

1. Determine the asymptote $(y = q)$
2. Determine the y- and x-intercepts
3. Determine the shape
4. Use table method to plot at least 2 other points

EXAMPLE:
Sketch the following graph showing all intercepts with the axes and asymptotes: $f(x) = 3^x$

SOLUTION:

Asymptote: $y = 0$

y-intercept: $y = 3^0 = 1$

x-intercept: none as $y = 0$ asymptote

Shape:

$a > 0, b > 1$

Increasing function

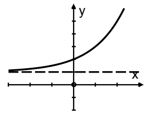
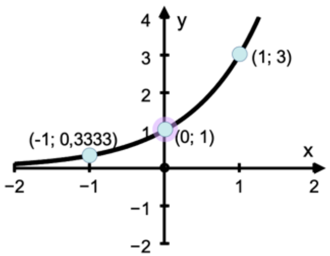


Table:

x	–1	1
y	$\frac{1}{3}$	3



Domain: $x \in \mathbb{R}$

Range: $y \in (0;\infty)$ *or* $y > 0$

EXAMPLE:
Sketch the following graph showing all intercepts with the axes and asymptotes:

$y = \left(\frac{1}{2}\right)^x - 2$

SOLUTION:

Asymptote: $y = -2$

y-intercept: $y = \left(\frac{1}{2}\right)^0 - 2 = -1$

x-intercept: $0 = \left(\frac{1}{2}\right)^x - 2$

$2 = 2^{-x}$

$\therefore x = -1$

Shape:

$y = +a \cdot b^x - q$

$0 < b < 1$

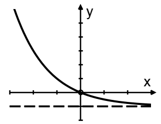
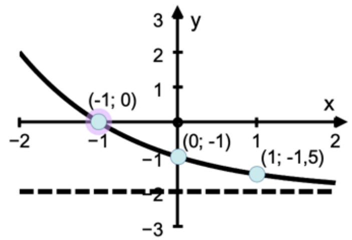


Table:

x	–1	1
y	0	1,5



Domain: $x \in \mathbb{R}$

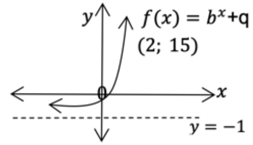
Range: $y \in (-2;\infty)$ *or* $y > -2$

FINDING THE EQUATION

STEPS:

1. First sub in the asymptote $y = q$
2. Sub in y-intercept to find a
3. Sub in other point to find b

EXAMPLE:
Determine the equation of:



Note $a = 1$ thus only need to find b and q

$f(x) = b^x - 1$ Sub in $(2;15)$

$15 = b^2 - 1$

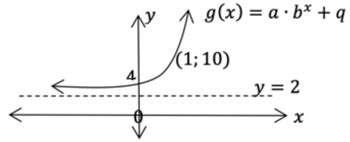
$16 = b^2$

$b = \pm 4$, but $b > 0,\ b \neq 1$

$\therefore b = 4$

$\therefore f(x) = 4^x - 1$

EXAMPLE:
Determine the equation of:



$g(x) = a \cdot b^x + 2$

$4 = a \cdot b^0 + 2$ Sub in $(0;4)$

$2 = a$

$g(x) = 2 \cdot b^x + 2$

$10 = 2 \cdot b^1 + 2$ Sub in $(1;10)$

$8 = 2 \cdot b$

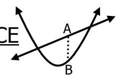
$4 = b$

$\therefore g(x) = 2 \cdot 4^x + 2$

FUNCTIONS AND GRAPHS

GRAPH APPLICATION

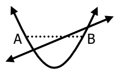
DISTANCE



Steps for determining VERTICAL DISTANCE

1. Determine the vertical distance
   Vertical distance = top graph – (bottom graph)
2. Substitute the given x-value to derive your answer

Steps for determining HORIZONTAL DISTANCE



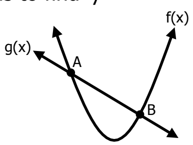
1. Find the applicable x-values
   $AB = x_B - x_A$ (largest – smallest)

NOTE:
- Distance is <u>always</u> positive
- Distance on a graph is measured in <u>units</u>

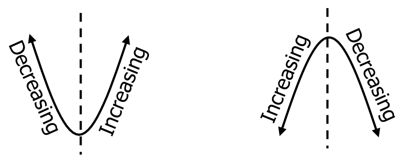
INTERSECTION OF GRAPHS

Steps for determining POINTS OF INTERSECTION

1. Equate the two functions
   $f(x) = g(x)$
2. Solve for x (look for the applicable x-value: A or B)
3. Substitute the applicable x-value into any of the two equations to find 'y'

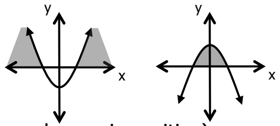


INCREASING/DECREASING



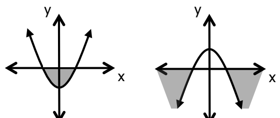
NOTATION

- $f(x) > 0$ ⊕
  (<u>above</u> the line y = 0)



(i.e. where y is positive)

- $f(x) < 0$ ⊖
  (<u>below</u> the line y = 0)



(i.e. where y is negative)

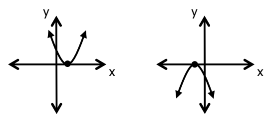
⊕ ⊖
- $f(x) \cdot g(x) \leq 0$ ⊖

⊖ ⊕
(one graph lies above y = 0 and one graph lies below y = 0)

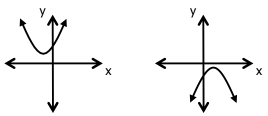
- $f(x) \geq g(x)$
  top bottom
  (i.e. f(x) lies <u>above</u> g(x))
- $f(x) = g(x)$
  (point of intersection)

ROOTS & PARABOLAS

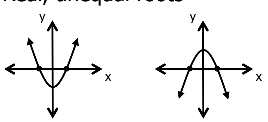
- Equal, real roots



- Non-real/ No real roots



- Real, unequal roots

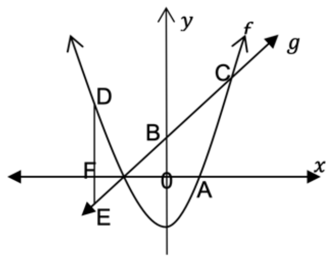


TRANSFORMATIONS OF GRAPHS

1. <u>Reflection in x-axis</u>: y becomes negative (i.e. all signs on right hand side of equation change).
2. <u>Reflection in y-axis</u>: all x's become negative
3. <u>Reflection in both axes</u>: both x and y become negative
4. <u>Horizontal Shift</u>: q changes, if up then add to q and if down subtract from q.

EXAMPLE:

Sketched are the graphs of $f(x) = x^2 - 1$ and $g(x) = x + 1$.



<u>QUESTIONS</u>:

a) the range of $f(x)$
b) equation of the axis of symmetry of $f(x)$
c) the coordinates of A
d) the length of OB
e) the coordinates of C
f) the length DE if OF is 4 units.
g) for which value(s) of x is
   i) $f(x) \geq 0$
   ii) $f(x) \cdot g(x) < 0$
   iii) $f(x)$ decreasing
h) Give the equation of $h(x)$ formed if $g(x)$ is reflected in the y-axis.
i) Give the equation of $k(x)$ formed if $f(x)$ is translated 3 units up.

SOLUTIONS:

a) $y \in [-1; \infty)$
b) $x = 0$
c) x–intercept ($\therefore y = 0$)
   $0 = x^2 = 1$
   $0 = (x - 1)(x + 1)$
   $\therefore x = 1 \quad or \quad x = -1$
   $\therefore A(1; 0)$
d) B is y–intercept $\therefore OB = 1$ unit
e) $f(x) = g(x)$ at C
   $x^2 - 1 = x + 1$
   $x^2 - x - 2 = 0$
   $(x - 2)(x + 1) = 0$
   $x = 2 \ or \ x = -1$
   $y = 2 + 1$ n/a
   $y = 3$
   $\therefore A(2; 3)$
f) OF on left side of x-axis $\therefore x = -4$
   $DE = f(-4) - g(-4)$
   $DE = ((-4)^2 - 1) - ((-4) + 1)$
   $DE = 15 - (-3)$
   $DE = 18$ units
g) i. $x \in (-\infty; -1] \ or \ [1; \infty)$
   ii. $x \in (-\infty; -1) \ or \ (-1; 1)$
   alt. $x \in (-\infty; 1); x \neq -1$
   iii. $x \in (-\infty; 0)$
h) $h(x) = -x + 1$
i) $k(x) = x^2 - 1 + 3$
   $k(x) = x^2 + 2$

PROBABILITY

TERMINOLOGY

Outcome: Result of an experiment.

Event: An event is a collection of outcomes that satisfy a certain condition. An event is denoted with the letter E and the number of outcomes in the event with n(E).

Dependent events: When the first event (A) affects the other's outcomes. E.g. choosing two coloured marbles from a bag, with replacement, thus, the first choice doesn't affect the outcome of the second choice.

Independent event: Events that do not affect each other's outcomes.

Cards in a deck: 52

Suits in a deck: 4

Specific card: 4

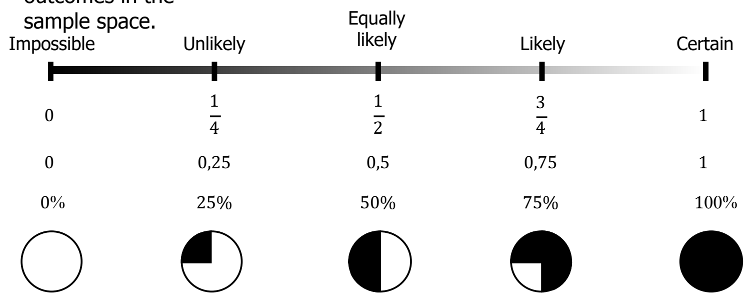
Sample spaces: All possible outcomes of the experiment. E.g. rolling a dice S={1;2;3;4;5;6}. Intersection of sets.

Unbiased: All events are equally likely to happen.

Complimentary event: Those two mutually exclusive events whose sum of probabilities equal to 1.

PROBABILITY

Likelihood of an event happening The probability of an event is the ratio between the number of outcomes in the event set and the number of possible outcomes in the sample space.



The ocean turns into milkshake

A coin dropped will land on heads

Theoretical Probability of an event happening:

$S = \{\text{sample set}\}$

$A = \{\text{event A}\}$

$B = \{\text{event B}\}$

$A \cup B = \{\text{A union B}\}$ = in sets A or B

$A \cap B = \{\text{A intersection B}\}$ = in sets A and B

Theoretical Probability of an event happening:

$$P(E) = \frac{\text{number of possible times event can occur}}{\text{number of possible outcomes}} = \frac{n(E)}{n(S)}$$

E = event S = sample space

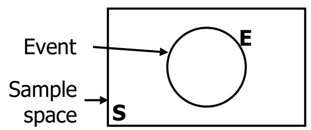
Relative frequency or Experimental probability:

$$P(E) = \frac{\text{number of times the event occured}}{\text{number of trials done}}$$

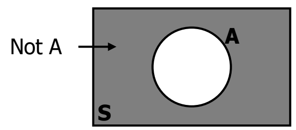
Addition Rule (OR/+):

P(A or B) = P(A) + P(B) − P(A and B) **or**
P(A and B) = P(A) + P(B) − P(A or B)

Probability using a Venn diagram



Compliment of an event

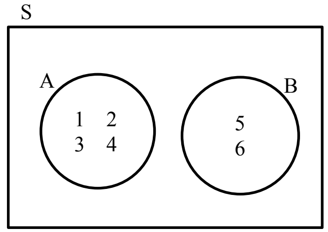


Complimentary Events:

Events A and B are **complimentary** events if they are mutually exclusive **and** exhaustive.

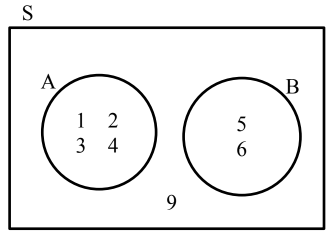
Exhaustive Events:

Events are **exhaustive** when they cover all elements in the sample set.



Mutually Exclusive:

A and B are **mutually exclusive** events as they have no elements in common.



NOTE:

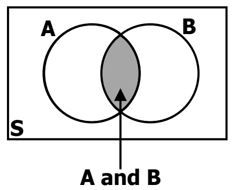
$n(A \cup B) = n(A) + n(B)$

$n(A \cap B) = 0$

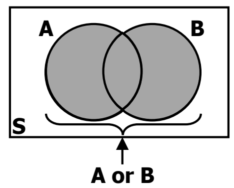
$P(A \cup B) = P(A) + P(B)$

Inclusive Events:

Intersecting events:

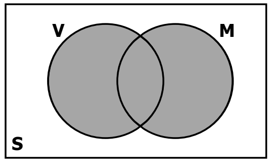


Union of events:

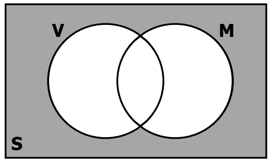


PROBABILITY

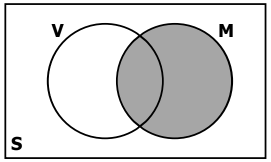
$V \cup M$



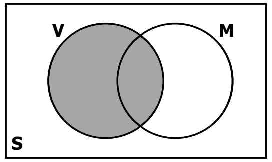
$(V \cup M)'$ or $V' \cap M'$



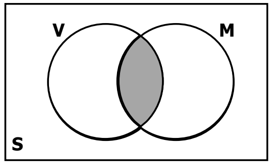
M



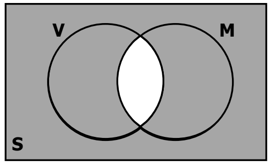
V



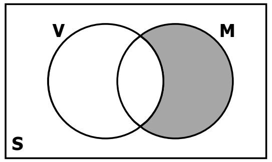
$V \cap M$



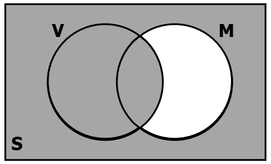
$(V \cap M)'$ or $V' \cup M'$



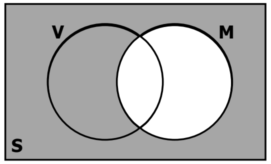
$V' \cap M$



$V \cup M'$



M'



EXAMPLE 1

A dice is rolled 100 times. It lands on 2 six-teen times. Calculate the relative frequency and compare this to the theoretical probability.

$$\text{Probability} = \frac{1}{6} = 0{,}1667$$

$$\text{Relative frequency} = \frac{\text{frequency of event}}{\text{number of trials}}$$
$$= \frac{16}{100}$$
$$= 0{,}16$$

The more an experiment is repeated the closer the relative frequency and the theoretical probability will be.

EXAMPLE 2

Elmari carried out a survey in her town to establish how many passengers travel in each vehicle. The following table shows her results:

Number of passengers	Number of cars
0	7
1	11
2	6
3	4
4	2

What is the probability that a vehicle has more than two passengers?

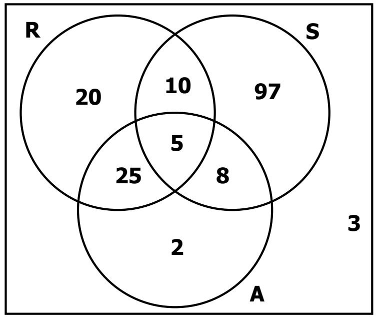
SOLUTION:
There are 30 vehicles in the survey, so n(S)=30. Let A be the event "cars with more than two passengers". This means that we only count the vehicles with three and four passengers. Therefore, n(A)=4+2=6.

$$P(A) = \frac{n(A)}{n(S)}$$
$$= \frac{6}{30}$$
$$= \frac{1}{5}$$

EXAMPLE 3

Questions:
Calculate from the Venn diagram for a grade 6 group in which the number of equally likely ways the events (Reading(R); Sports(S) and Art(A)) can occur has been filled in:

Gr 6



1. P(A∩R∩S)
2. P(R and A and not S)
3. P(A or R)
4. P(S or R and not A)

Solutions:

1. $P(A \cap R \cap S) = \frac{5}{170} = \frac{1}{34}$
2. $P(\text{R and A and not S}) = \frac{25}{170} = \frac{5}{34}$
3. $P(\text{A or R}) = \frac{70}{170} = \frac{7}{17}$
4. $P(\text{S or R and not A}) = \frac{127}{170}$

EXAMPLE 4

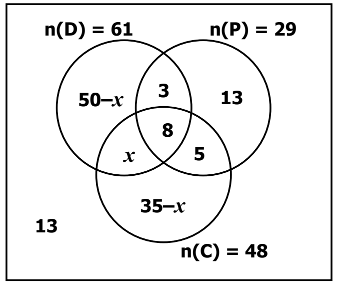
Questions:
120 Gr 12 girls at Girls High where asked about their participation in the school's culture activities:

- 61 girls did drama (D)
- 29 girls did public speaking (P)
- 48 girls did choir (C)
- 8 girls did all three
- 11 girls did drama and public speaking
- 13 girls did public speaking and choir
- 13 girls did no culture activities

1. Draw a Venn diagram to represent this information.
2. Determine the number of Girls who participate in drama and choir only.
3. Determine the probability that a grade 12 pupil chose at random will:
   a. only do choir.
   b. not do public speaking.
   c. participate in at least two of these activities.

Solutions:

1. **n(S) = 120**

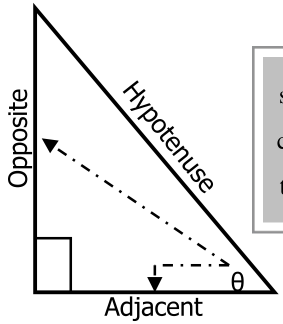


2. $(50 - x) + 3 + 13 + x + 8 + 5 + (35 - x) + 13 = 120$
   $127 - x = 120$
   $\therefore x = 7$
   $\therefore 50 - x = 43$ and $35 - x = 28$

3.
   a. $P(\text{C only}) = \frac{35}{120} = \frac{28}{120} = 0{,}23$
   b. $P(P') = \frac{120 - 29}{120} = \frac{91}{120} = 0{,}76$
   c. $P(\text{at least 2}) = \frac{3 + 7 + 8 + 5}{120} = \frac{23}{120} = 0{,}19$

TRIGONOMETRY

BASIC DEFINITIONS



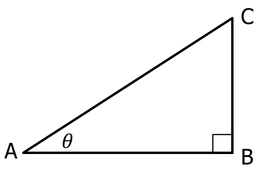
$\sin\theta = \frac{o}{h}$ soh

$\cos\theta = \frac{a}{h}$ cah

$\tan\theta = \frac{o}{a}$ toa

These are our basic trig ratios.

EXAMPLE

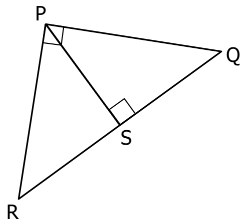


$\sin\theta = \frac{o}{h} = \frac{BC}{AC}$

$\cos\theta = \frac{a}{h} = \frac{AB}{AC}$

$\tan\theta = \frac{o}{a} = \frac{BC}{AB}$

EXAMPLE



1. Write down two ratios for $\cos R$

$\cos R = \frac{a}{h} = \frac{PR}{QR} = \frac{RS}{PR}$

(in $\triangle$ PQR) (in $\triangle$ PRS)

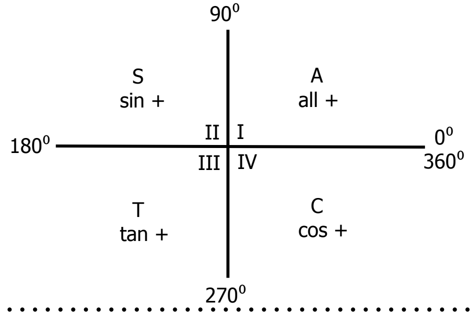
2. Write down two ratios for $\tan Q$

$\tan Q = \frac{o}{a} = \frac{PR}{PQ} = \frac{PS}{QS}$

(in $\triangle$ PQR) (in $\triangle$ PQS)

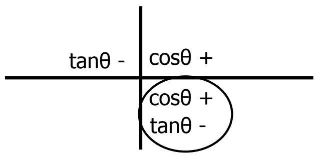
BASIC CAST DIAGRAM

Shows the quadrants where each trig ratio is +



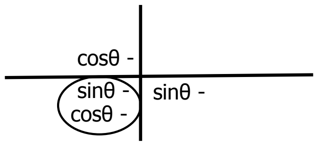
EXAMPLE

1. In which quadrant does θ lie if tanθ < 0 and cosθ > 0?



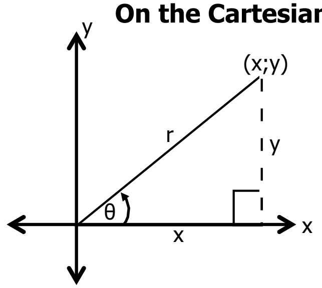
Quadrant IV

2. In which quadrant does θ lie if sinθ < 0 and cosθ < 0?



Quadrant III

On the Cartesian Plane



$\sin\theta = \frac{y}{r}$

$\cos\theta = \frac{x}{r}$

$\tan\theta = \frac{y}{x}$

Remember:

- $x^2 + y^2 = r^2$ (Pythagoras)
- Angles are measured upwards from the positive (+) x-axis (anti-clockwise) up to the hypotenuse (r).

PYTHAGORAS PROBLEMS

Steps:
1. Isolate the trig ratio
2. Determine the quadrant
3. Draw a sketch and use Pythagoras
4. Answer the question

EXAMPLE

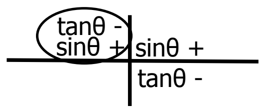
If 3sinθ – 2 = 0 and tanθ < 0, determine $\sin^2\theta + \cos^2\theta$ without using a calculator and by using a diagram.

Step 1:

$3\sin\theta - 2 = 0$

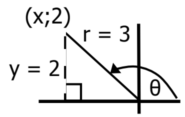
$\sin\theta = \frac{2}{3}$ $\frac{y}{r}$

Step 2:



$\therefore$ *Quadrant II*

Step 3:



$x^2 + y^2 = r^2$

$x^2 + (2)^2 = (3)^2$

$x^2 = 5$

$x = \pm\sqrt{5}$

$\therefore x = -\sqrt{5}$

Step 4: $\sin^2\theta + \cos^2\theta$

$= \left(\frac{2}{3}\right)^2 + \left(\frac{-\sqrt{5}}{3}\right)^2$

$= \frac{4}{9} + \frac{5}{9}$

$= \frac{9}{9}$

$= 1$

Remember:
$\sin\theta = \frac{y}{r}$
and
$\cos\theta = \frac{x}{r}$

RECIPROCALS

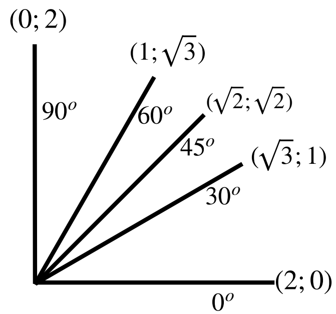
$\frac{1}{\sin\theta} = \text{cosec}\theta$ $\left(\frac{r}{y}\right)$ or $\left(\frac{h}{o}\right)$

$\frac{1}{\cos\theta} = \sec\theta$ $\left(\frac{r}{x}\right)$ or $\left(\frac{h}{a}\right)$

$\frac{1}{\tan\theta} = \cot\theta$ $\left(\frac{x}{y}\right)$ or $\left(\frac{a}{o}\right)$

Special Angles

$r = 2$ $(x; y)$



EXAMPLE

Simplify without the use of a calculator:

1. $\tan 60° + \cot 30°$

$= \left(\frac{\sqrt{3}}{1}\right) + \left(\frac{\sqrt{3}}{1}\right)$

$= \sqrt{3} + \sqrt{3}$

$= 2\sqrt{3}$

2. $\sin^2 45° - \cos^2 30°$

$= \left(\frac{\sqrt{2}}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$

$= \frac{2}{4} - \frac{3}{4}$

$= -\frac{1}{4}$

TRIGONOMETRY

TRIG EQUATIONS

For $0° < \theta < 90°$

Steps:

1. Isolate the trig ratio
2. Reference angle (shift on the calculator)
3. Solve for θ

REMEMBER:
Only round off at the **END**

EXAMPLES

Solve for θ

1. $3\sin\theta - 1 = 0$

$\sin\theta = \dfrac{1}{3}$

Ref $\angle$: $19{,}47°$

$\therefore \theta = 19{,}47°$

2. $\tan(3\theta + 30°) - 1 = 0$

$\tan(3\theta + 30°) = 1$

Ref $\angle$: $45°$

$\therefore 3\theta + 30° = 45°$

$3\theta = 15°$

$\theta = 5°$

3. $\sec 2\theta = 2$

$\dfrac{1}{\cos 2\theta} = 2$

$\therefore \cos 2\theta = \dfrac{1}{2}$

Ref $\angle$: $60°$

$\therefore 2\theta = 60°$

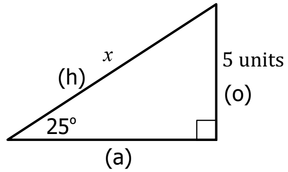
$\theta = 30°$

SOLVING RIGHT-ANLGED TRIANGLES

* Missing Sides *

EXAMPLES

1. Calculate x



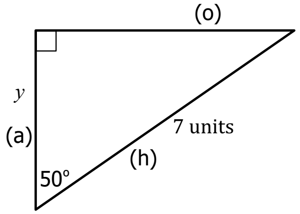
$\sin 25° = \dfrac{o}{h} = \dfrac{5}{x}$

$\therefore x \sin 25° = 5$

$x = \dfrac{5}{\sin 25°}$

$x = 11{,}83$ units

2. Calculate y



$\cos 50° = \dfrac{a}{h} = \dfrac{y}{7}$

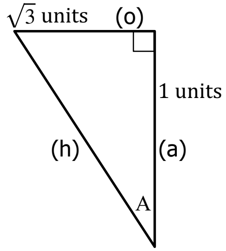
$7\cos 50° = y$

$y = 4{,}50$ units

* Missing Angles *

EXAMPLES

1. Calculate $\hat{A}$



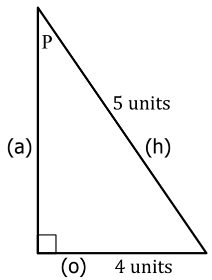
$\tan A = \dfrac{o}{a} = \dfrac{\sqrt{3}}{1}$

$\tan A = \sqrt{3}$ * [shift] [tan] [$\sqrt{3}$]

Ref $\angle$: $60°$

$\therefore \hat{A} = 60°$

2. Calculate $\hat{P}$



$\sin \hat{P} = \dfrac{o}{h} = \dfrac{4}{5}$

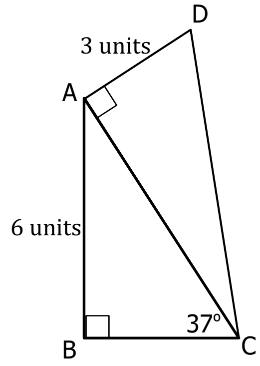
$\sin P = \dfrac{4}{5}$ * [shift] [sin] [4/5]

Ref $\angle$: $53{,}13°$

$\therefore \hat{P} = 53{,}13°$

PROBLEM SOLVING

Determine the area of quadrilateral ABCD.



In $\triangle$ ABC :

$\tan 37° = \dfrac{o}{a} = \dfrac{6}{BC}$

$BC = \dfrac{6}{\tan 37°}$

$\therefore BC = 7{,}96$ units ← Don't round off early. Remember to store answer in calculator

$\therefore AC^2 = AB^2 + BC^2$ (Pythag)

$AC^2 = (6)^2 + (7{,}96)^2$ ← Use the 'unrounded' answer from your calculator

$AC^2 = 99{,}3977$

$\therefore AC = 9{,}97$ units

Area of $\triangle$ ABC :

$\text{Area} = \dfrac{1}{2} b \cdot \perp h$

$= \dfrac{1}{2}(BC)(AB)$

$= \dfrac{1}{2}(7{,}96)(6)$

$= 23{,}89 \text{ units}^2$

Area of $\triangle$ ACD :

$\text{Area} = \dfrac{1}{2} b \cdot \perp h$

$= \dfrac{1}{2}(AC)(AD)$

$= \dfrac{1}{2}(9{,}97)(3)$

$= 14{,}95 \text{ units}^2$

Area of Quad ABCD :

$\text{Area} = 23{,}89 + 14{,}95$

$= 38{,}84 \text{ units}^2$

TRIG GRAPHS

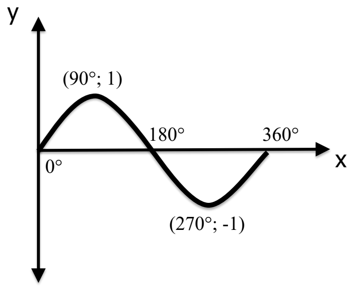
IMPORTANT!

When sketching trig graphs, you need to label the following:

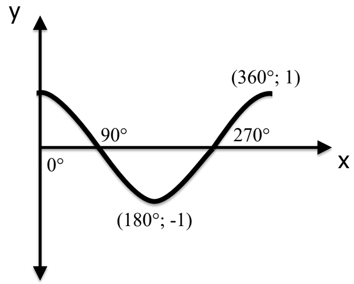
- **both axes**
- **x- and y-intercepts**
- **turning points**
- **endpoints** (if not on the axes)
- **asymptotes** (tan graph only)

BASICS

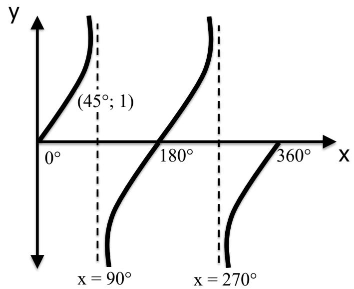
- $y = \sin x$ for $x \in [0°; 360°]$



- $y = \cos x$ for $x \in [0°; 360°]$



- $y = \tan x$ for $x \in [0°; 360°]$



Notes for $\sin x$ and $\cos x$:

- ❖ Key points (intercepts/turning pts) every 90°
- ❖ Period (1 complete graph): 360°
- ❖ Amplitude (halfway between min and max): 1

Notes for $\tan x$:

- ❖ Key points every 45°
- ❖ Period (1 complete graph): 180°
- ❖ No amplitude can be defined
- ❖ Asymptotes at $x = 90° + k180°, k \in \mathbb{Z}$

AMPLITUDE CHANGE

- $y = a \cdot \sin x$ or $y = a \cdot \cos x$ or $y = a \cdot \tan x$

If a > 1 : stretch
0 < a < 1 : compress
a < 0 : reflection in x-axis

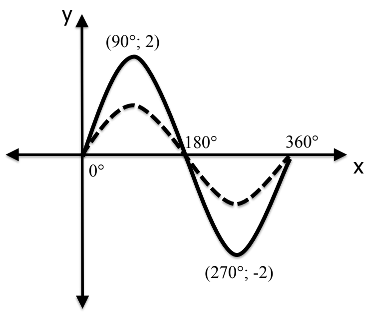
EXAMPLES

1. $y = 2\sin x$
(solid line)

$y = \sin x$
(dotted line - for comparison)

* Amplitude = 2

* Range: $y \in [-2; 2]$

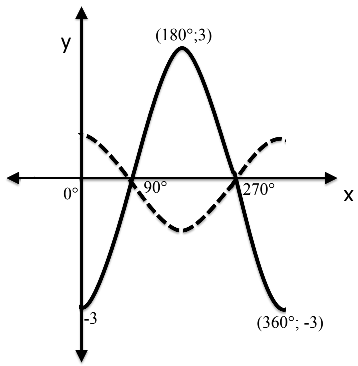


2. $y = -3\cos x$
(solid line)

$y = \cos x$
(dotted line - for comparison)

* Amplitude = 3

* Range: $y \in [-3; 3]$



VERTICAL SHIFT

- $y = \sin x + q$ or $y = \cos x + q$ or $y = \tan x + q$

If q > 0 : upwards (e.g: $y = \sin x + 1$)
If q < 0 : downwards (e.g: $y = \cos x - 2$)

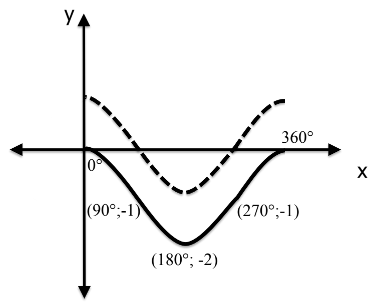
EXAMPLE

$y = \cos x - 1 \quad x \in [0°; 360°]$ (solid line)

$y = \cos x$ (dotted line - for comparison)

* Amplitude = 1

* Range: $y \in [-2; 0]$



MIXED EXAMPLE

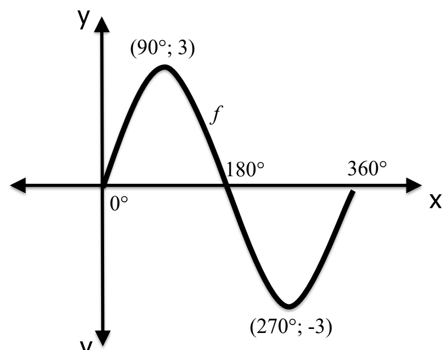
Given $f(x) = 3\sin x$ and $g(x) = \cos x - 2$

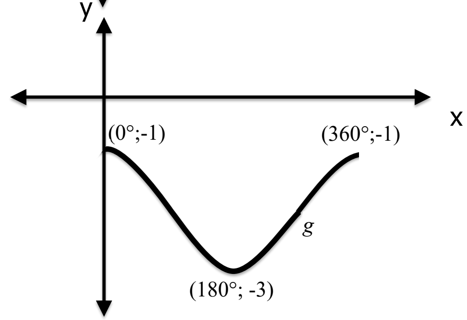
Questions:

1. What is the period of f?
2. State the amplitude of f?
3. Give the amplitude of g?
4. For $x \in [0°; 360°]$ sketch f and g on separate sets of axes.
5. Use the graphs to determine the values of x for which:
   a. $f(x) > 0$
   b. $g(x)$ is increasing
6. Explain, in words, the transformation that takes $g(x) = \cos x - 2$ to $y = 2\cos x + 2$

Solutions:

1. $360°$
2. 3
3. 1
4.





5.
   a. $x \in (0°; 180°)$
   b. $x \in (180°; 360°)$

Remember:
A graph is **positive** when it is **above the x-axis**

Remember:
A graph is **increasing** when **the gradient is positive**

6. $g(x) = (1)\cos x - 2$

to

$y = 2\cos x + 2$

double the amplitude — move 4 units up

∴ g is stretched by a factor of 2 and translated 4 units up.

EUCLIDEAN GEOMETRY

Grade 8 and 9 Revision

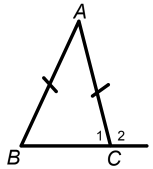
Deductive logic in geometry

Working reasoning to conclude next answer in known as deductive logic.

Examples:

1. If $a = b$ and $b = c$ then $a = c$
2. If $x = y$ and $p = q$ then $x + p = y + q$
3. If $\hat{P} - \hat{Q} = 180°$ and $\hat{S} - \hat{Q} = 180°$ then: $\hat{P} = \hat{S}$

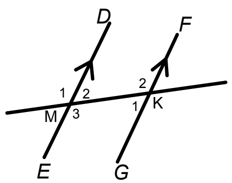
THEORY TO REMEMBER



$\hat{B} = \hat{C}_1$ (∠'s opp. = sides)

$\hat{A} + \hat{B} + \hat{C}_1 = 180°$ (sum ∠'s of Δ)

$\hat{C}_2 = \hat{A} + \hat{B}$ (ext. ∠'s of Δ)



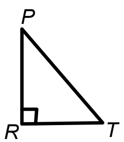
$\hat{K}_2 = \hat{M}_1$ (corres. ∠'s DE//GF)

$\hat{K}_2 = \hat{M}_3$ (alt. ∠'s DE//GF)

$\hat{K}_2 + \hat{M}_2 = 180°$ (co-int. ∠'s DE//GF)

$\hat{M}_1 = \hat{M}_3$ (vert. opp. ∠'s)

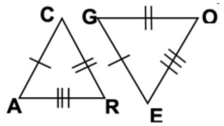
$\hat{K}_2 + \hat{K}_1 = 180°$ (∠'s on a str. line)



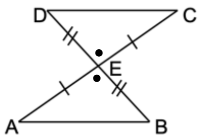
$PT^2 = PR^2 + RT^2$ (Pythag. Th.)

CONGRUENT TRIANGLES

Remember there are four reasons for congruency and the triangles must be written in order of equal parts.

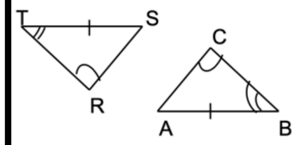


ΔACR ≡ ΔEGO (SSS)

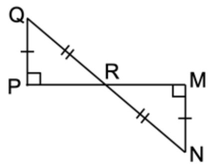


ΔDEC ≡ ΔBEA (SAS)

NOTE: must be the INCLUDED angle



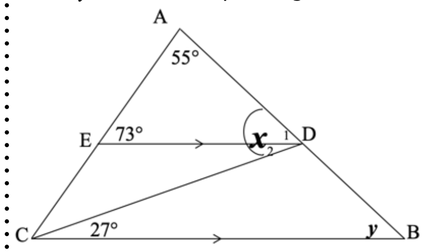
ΔRST ≡ ΔCAB (SAA)



ΔPQR ≡ ΔMNR (RHS)

EXAMPLE:

If $A\hat{D}C = x$, find with reasons, the size of angles x and y. Show all steps and give all reasons.



SOLUTION:

$\hat{D}_2 = 27°$ (alt. ∠'s DE//CB)

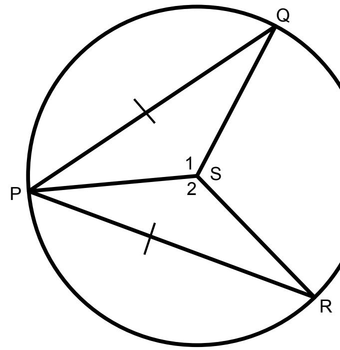
$\hat{D}_1 = 52°$ (sum ∠'s of Δ)

$\therefore x = 79°$

$y = \hat{D}_1 = 52°$ (corres. ∠'s, DE//CB)

EXAMPLE:

Given $PQ = PR$ and circle centre S. Prove that PS bisects angle $Q\hat{S}R$.



SOLUTION:

In ΔPSQ and ΔPSR

PS = PS (common)

PQ = PR (given)

SQ = SR (radii)

△ PSQ ≡ △ PSR (SSS)

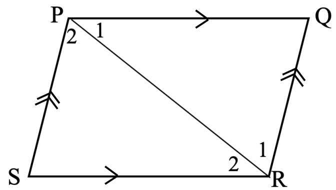
$\therefore \hat{S}_1 = \hat{S}_2$ (≡ Δ's)

EXAMPLE:

Given $PQ \parallel RS$ and $PS \parallel QR$, prove:

a) $PQ = RS$

b) $PS = QR$



SOLUTION:

a) In ΔPQR and ΔRSP

$\hat{P}_1 = \hat{R}_2$ (alt ∠'s, PQ ∥ RS)

PR = PR (common)

$\hat{R}_1 = \hat{P}_2$ (alt ∠'s, PS ∥ QR)

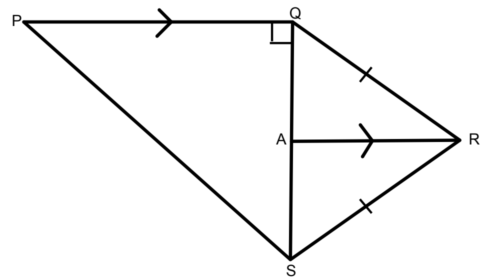
∴ ΔPRQ ≡ ΔRPS (SAA)

∴ PQ = RS (≡ △'s)

b) PS = QR(≡ Δ's)

EXAMPLE:

Given $P\hat{Q}S = 90°$, $QR = RS$ and $PQ \parallel AR$. Prove that $AQ = AS$.



SOLUTION:

In ΔARQ and ΔARS

AR = AR (common)

$P\hat{Q}R = Q\hat{A}R = 90°$ (corres. ∠'s, PQ ∥ AR)

QR = RS (given)

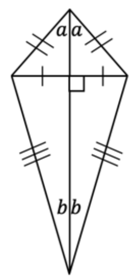
∴ ΔARQ ≡ ΔARS (RHS)

∴ AQ = AS (≡ Δ's)

PROPERTIES OF QUADRILATERALS

*For ALL quadrilaterals: sum of interior angles is 360°

1. KITE



- Two pairs of adjacent sides are equal.
- The longest diagonal bisects the angles.
- One diagonal is the perpendicular bisector of the other.
- One pair of opposite angles are congruent.
- Two pairs of adjacent sides are equal.
- The longest diagonal bisects the angles.
- One diagonal is the perpendicular bisector of the other.
- One pair of opposite angles are congruent.

2. TRAPEZIUM

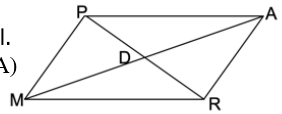
- One pair of opposite sides parallel.

EUCLIDEAN GEOMETRY

PROPERTIES OF QUADRILATERALS (CONT.)

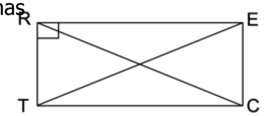
3. PARALLELOGRAM (parm)

- The opposite sides are parallel by definition. (PA ∥ RM & PM ∥ AR)
- The opposite sides are equal. PA = RM & PM = AR)
- The opposite angles are are equal. ($P\hat{A}R = P\hat{M}R$ & $M\hat{P}A = M\hat{R}A$)
- The diagonals bisect each other. (PD = DR & AD = DM)
- One pair of opposite sides parallel and equal.



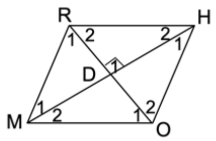
4. RECTANGLE (rect)

- Is a specialised parallelogram so has all the parallelogram properties.
- Diagonals are equal length. (RC = ET)
- Interior angles are each 90°.



5. RHOMBUS (rhom)

- Is a specialised parallelogram so has all the parallelogram properties.
- Adjacent sides are equal. (RH = HO = OM = MR)
- Diagonals are perpendicular to each other. ($\hat{D}_1 = 90°$)
- Diagonals bisect the angles. ($\hat{R}_1 \hat{=} \hat{R}_2$; $\hat{H}_1 = \hat{H}_2$; $\hat{O}_1 = \hat{O}_2$; $\hat{M}_1 = \hat{M}_2$)

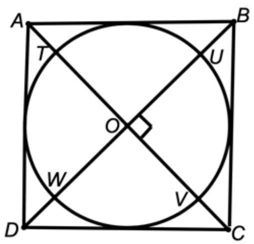


6. SQUARE (squ.)

- Is a specialised parallelogram, rectangle and rhombus so has all their properties.

EXAMPLE:

AT = BU = CV = DW. O is the centre of the circle TUVW and $B\hat{O}C = 90°$. AC and BD are straight lines. Prove that ABCD is a square.

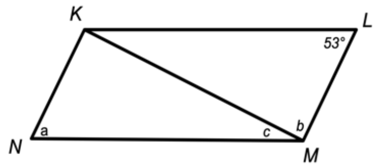


SOLUTION:

OT = OU = OV = OW (radii)
AT = BU = CV = DW (given)
∴ AO = BO = CO = DO
AC ⊥ DB (given)
∴ ABCD a squ. (diag. = and ⊥)

EXAMPLE:

KLMN is a parallelogram with KM = KL.



Determine the values of a, b and c.

SOLUTION:

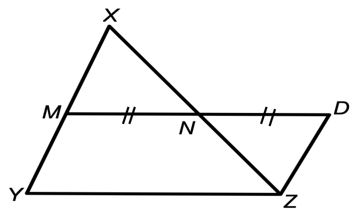
a = 53° (opp ∠'s parm)
b = 53° (∠'s opp = sides)
$M\hat{K}L$ = 74° (sum ∠'s of Δ)
c = 74° (alt ∠'s, KL ∥ NM)

EXAMPLE:

M and N are the midpoints of XY and XZ. MN is produced its own length to D.

Prove that:

a) XMZD a parallelogram
b) MYZD a parallelogram



SOLUTION:

a) XN = NZ (given mid pt)
MN = ND (given)
∴ XMZD a parm.
(diag bisect ea. other)

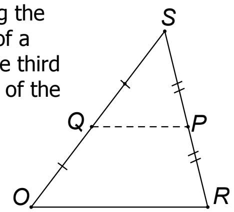
b) XY ∥ DZ (opp. sides parm XMZD)
XM = MY (given mid pt.)
XM = DZ (opp. sides parm XMZD)
∴ MY = DZ
∴ MYZD a parm.
(1 pr. opp. sides = & ∥)

MIDPOINT THEOREM

Mid-Point Theorem:

(mid-pt. Th.)

The line segment joining the midpoints of two sides of a triangle, is parallel to the third side and half the length of the third side.

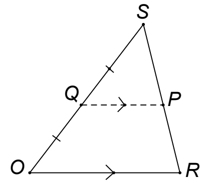


Therefore if $SQ = QO$ and $SP = PR$ then $PQ \parallel OR$ and $QP = \frac{1}{2}OR$ (mid-pt. Th.)

Converse:

(conv. mid-pt. Th.)

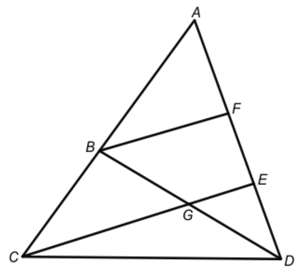
The line passing through the midpoint of one side of a triangle and parallel to another side, bisects the third side. The line is also equal to half the length of the side it is parallel to.



Therefore if $SQ = QO$ and $PQ \parallel OR$ then $SP = PR$ and $QP = \frac{1}{2}OR$ (conv. mid-pt. Th.)

EXAMPLE

In ΔACE, AB = BC, GE = 15 cm and AF = FE = ED.



Determine the length of CE.

SOLUTION:

In ΔACE:
AB = BC and AF = FE (given)
∴ BF ∥ CE and $BF = \frac{1}{2}CE$ (mid – pt. Th.)

In ΔDFB
FE = ED (given)
BF ∥ GE (proven)
∴ BG = GD and $GE = \frac{1}{2}BF$ (conv. mid – pt. Th.)
∴ BF = 2GE
∴ BF = 2(15) = 30 cm

CE = 2BF (proven)

HINTS WHEN ANSWERING GEOMETRY QUESTIONS

- Read the given information and mark on to the diagram if not already done.
- Never assume anything. If not given or marked on diagram is not true unless proved.
- As you prove angles equal or calculate angles mark them on to the diagram and write down statement and reason there and then.
- Make sure that by the end of the question you have used all the given information.
- If asked to prove something, it is true. **For example:** if asked to prove ABCD a parallelogram, it is a parallelogram. If you can't prove it, you can still use it as a parallelogram in the next part of the question.

ANALYTICAL GEOMETRY

What is Analytical Geometry?

(Co-ordinate Geometry): Application of straight line functions in conjunction with Euclidean Geometry by using points on a Cartesian Plane.

DISTANCE BETWEEN TWO POINTS

The distance between two points $(x_1; y_1)$ and $(x_2; y_2)$ is given by:

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

EXAMPLES

1. Determine the length of PQ if $P(-1; 4)$ and $Q(4; -2)$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$
$$= \sqrt{(-1-4)^2 + (4-(-2))^2}$$
$$= \sqrt{61}$$
$$= 7{,}81$$

2. If $A(1; 2)$, $B(-1; -5)$ and $C(x; -7)$ and $AB = BC$, calculate x

$$AB = \sqrt{(1+1)^2 + (2+5)^2}$$
$$= \sqrt{53}$$
$$BC = \sqrt{(x+1)^2 + (-7+5)^2}$$
$$= \sqrt{(x+1)^2 + 4}$$

but $AB = BC$

$$\therefore \sqrt{53} = \sqrt{(x+1)^2 + 4}$$
$$\therefore (x+1)^2 + 4 = 53$$
$$(x+1)^2 = 49$$
$$x + 1 = \pm 7$$

$x = -1 + 7$ or $x = -1 - 7$

$x = 6$ or $x = -8$

MIDPOINT OF A LINE SEGMENT

The midpoint between $(x_1; y_1)$ and $(x_2; y_2)$ is given by:

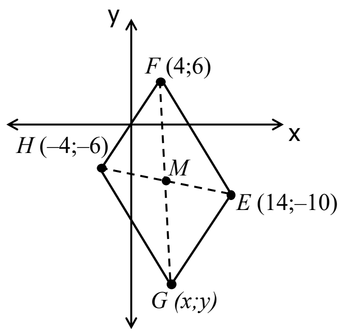
$$M(x; y) = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$$

EXAMPLES

1. Determine the midpoint of $P(-1; 4)$ and $Q(4; -2)$

$$\text{Midpnt} = \left(\frac{x_2 + x_1}{2}; \frac{y_2 + y_1}{2}\right)$$
$$= \left(\frac{-1+4}{2}; \frac{4-2}{2}\right)$$
$$= \left(\frac{3}{2}; 1\right)$$

2. $FEGH$ is a parallelogram. Calculate the co-ordinates of G.



Remember: in a parallelogram the diagonals bisect each other, $\therefore$ M is the mid-point of FG and EH.

$$\text{Midpnt of } EH = \left(\frac{-4+14}{2}; \frac{-6-10}{2}\right)$$
$$= (5; -8)$$
$$\text{Midpnt of } FG = \left(\frac{x+4}{2}; \frac{y+6}{2}\right)$$

$\therefore \frac{x+4}{2} = 5$ and $\frac{y+6}{2} = -8$

$x + 4 = 10$ $\quad$ $y + 6 = -16$

$x = 6$ and $y = -22$

$\therefore G(6; -22)$

GRADIENT OF A LINE

The gradient of a straight line between $(x_1; y_1)$ and $(x_2; y_2)$ is given by:

$$m = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

REMEMBER:

- Parallel (||) lines: $m_1 = m_2$
- Perpendicular (⊥) lines: $m_1 \times m_2 = -1$
- Horizontal (–) lines $[y = c]$: $m = 0$
- Vertical (|) lines $[x = c]$: m is undefined

EXAMPLE

Given $A(2; 3)$ and $B(-3; 1)$.

1. Determine the gradient of the line AB

$$m_{AB} = \left(\frac{y_2 - y_1}{x_2 - x_1}\right) = \left(\frac{3-1}{2+3}\right) = \left(\frac{2}{5}\right)$$

2. Determine the gradient of the line parallel to AB

$$m_{AB} = m_{\parallel} = \frac{2}{5}$$

For || lines: $m_1 = m_2$

3. Determine the gradient of the line perpendicular to AB

For ⊥ lines: Flip the fraction and change the sign

$$m_{AB} \times m_{\perp} = -1$$
$$\frac{2}{5} \times m_{\perp} = -1$$
$$\therefore m_{\perp} = -\frac{5}{2}$$

COLLINEAR POINTS

Points on the same line, hence, gradients between the points are equal.

EXAMPLE

If $T(5; 2)$, $U(7; 4)$ and $V(b; -5)$ are collinear, calculate the value of b.

Collinear $\therefore m_{TU} = m_{UV}$

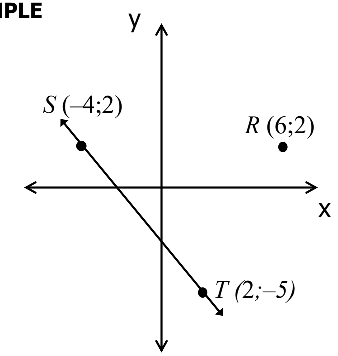
$$\frac{2-4}{5-7} = \frac{4+5}{7-b}$$
$$1 = \frac{9}{7-b}$$
$$7 - b = 9$$
$$b = -2$$

EQUATIONS OF STRAIGHT LINES

$$y = mx + c$$

gradient (m) y-intercept (c)

EXAMPLE



Determine the equation of the line:

1. ST

$m_{ST} = \frac{2+5}{-4-2} = -\frac{7}{6}$

$y = -\frac{7}{6}x + c$

Sub in $S(-4;2)$ (or T)

$2 = -\frac{7}{6}(-4) + c$

$c = -\frac{8}{3}$

$y = -\frac{7}{6}x - \frac{8}{3}$

2. SR

Horizontal line

$S(-4;2)$ and $R(6;2)$ have the same y-value.

$\therefore m = 0$

$\therefore y = 2$

3. parallel to ST, through point R.

$m_{ST} = m_{\parallel} = -\frac{7}{6}$

$\therefore y = -\frac{7}{6}x + c$

Sub in $R(6;2)$

$2 = -\frac{7}{6}(6) + c$

$c = 9$

$y = -\frac{7}{6}x + 9$

4. perpendicular to ST, through point T.

$m_{ST} \times m_{\perp} = -1$

$-\frac{7}{6} \times m_{\perp} = -1$

$\therefore m_{\perp} = \frac{6}{7}$

$y = \frac{6}{7} + c$

Sub in $T(2;-5)$

$-5 = \frac{6}{7}x + c$

$c = \frac{-47}{7}$

$\therefore y = \frac{6}{7}x - \frac{47}{7}$

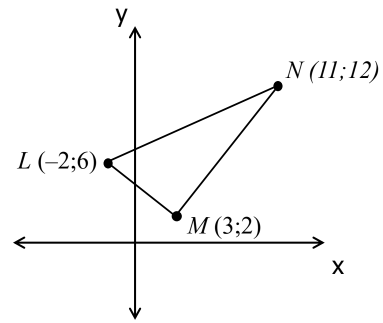
5. perpendicular to SR, through S.

$\perp$ to horizontal line is a vertical line through $(-4;2)$

$\therefore x = -4$

MIXED EXAMPLE 1

Prove that $\triangle LMN$ is right-angled



$m_{LM} = \frac{6-2}{-2-3} = -\frac{4}{5}$

$m_{NM} = \frac{12-2}{11-3} = \frac{5}{4}$

$\therefore m_{LM} \times m_{NM}$

$= -\frac{4}{5} \times \frac{5}{4}$

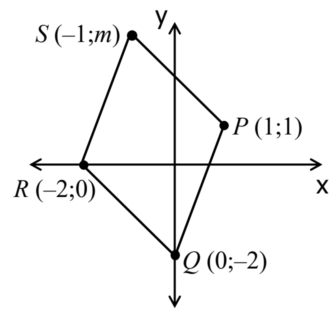
$= -1$

$\therefore LM \perp MN$

$\therefore \triangle LMN$ is right-angled

MIXED EXAMPLE 2

Quadrilateral $PQRS$ is given



1. Determine the length of PQ

$PQ = \sqrt{(1-0)^2 + (1+2)^2}$

$= \sqrt{10}$

2. Find the gradient of RQ

$m_{RQ} = \frac{0+2}{-2-0} = -1$

NOTE:
There are 5 ways to prove a quad is a parm

1. both pairs of opposite sides equal
2. both pairs of opposite sides parallel
3. one pair of opposite sides equal and parallel
4. diagonals bisect each other
5. both pairs of opposite angles equals

3. If $RQ \parallel SP$ determine the value of m

$m_{RQ} = m_{SP}$

$-1 = \frac{m-1}{-1-1}$

$2 = m - 1$

$\therefore m = 3$

4. Prove that $PQRS$ is a parallelogram

* You could use methods 1-4 to answer this question. Let's use 4 this time (diags bisect)

Midpt $PR = \left(\frac{1-2}{2}; \frac{1-0}{2}\right) = \left(-\frac{1}{2}; \frac{1}{2}\right)$

Midpt $QS = \left(\frac{-1-0}{2}; \frac{3-2}{2}\right) = \left(-\frac{1}{2}; \frac{1}{2}\right)$

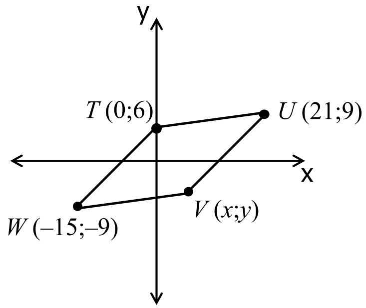
$\therefore$ Midpt PR = Midpt QS

$\therefore$ Diags bisect each other

$\therefore PQRS$ is a parm

MIXED EXAMPLE 3

Parallelogram $TUVW$ is given



1. Determine the gradient of UW

$m_{UW} = \frac{9+9}{21+15} = \frac{1}{2}$

2. Determine (by inspection) the co-ordinates of V

$T \rightarrow U \quad x: 0 \rightarrow 21 \quad \therefore x + 21$

$y: 6 \rightarrow 9 \quad \therefore y + 3$

$\therefore W \rightarrow V \quad x + 21: -15 + 21 = 6$

$y + 3: -9 + 3 = -6$

$TU \parallel WV$

$\therefore V(6; -6)$

3. Calculate the length of TW (in simplest surd form)

$TW = \sqrt{(-15-0)^2 + (-9-6)^2}$

$= 15\sqrt{2}$

4. Prove that $\triangle TUW$ is isosceles

$TU = \sqrt{(0-21)^2 + (6-9)^2}$

$= 15\sqrt{2}$

$\therefore TW = TU$

$\therefore \triangle TUW$ is isosceles

5. Hence, what type of parm is $TUVW$? Give a reason.

Rhombus. Parm with adjacent sides equal.

6. Determine the equation of the line perpendicular to UW and passing through point W

$m_{UW} \times m_{\perp} = -1$

$\frac{1}{2} \times m_{\perp} = -1$

$\therefore m_{\perp} = -2$

$y = -2x + c$

Sub in $W(-15:-9)$

$-9 = -2(-15) + c$

$c = -39$

$\therefore y = -2x - 39$

7. If U, $R(3;k)$ and W re collinear, find the value of k

Collinear $\therefore m_{UR} = m_{UW}$

$\frac{k-9}{3-21} = \frac{1}{2}$

$\frac{k-9}{-18} = \frac{1}{2}$

$\therefore k - 9 = -9$

$\therefore k = 0$

STATISTICS

REMINDER

Discrete data: Data that can be counted, e.g. the number of people.

Continuous data: quantitative data that can be measured, e.g. temperature range.

Measures of central tendency: a descriptive summary of a dataset through a single value that reflects the centre of the data distribution.

Measures of dispersion: The dispersion of a data set is the amount of variability seen in that data set.

Outliers: Any data value that is more than 1,5 IQR to the left of Q_1 or the right of Q_3, i.e.
Outlier < Q_1 – (1,5×IQR) or
Outlier > Q_3 + (1,5×IQR)

NB: Always arrange data in ascending order.

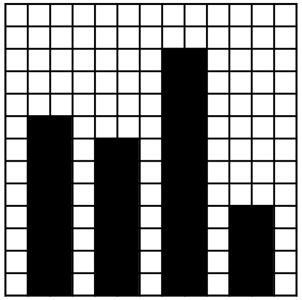
FREQUENCY TABLE

Mark	Tally	Frequency
4	||	2
5	||	2
6	||||	4
7	~~||||~~	5
8	||||	4
9	||	2
10	|	1

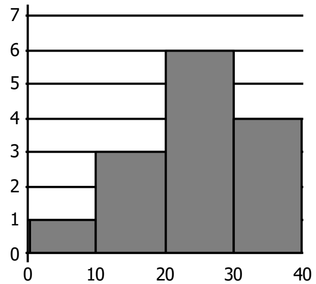
STEM AND LEAF PLOTS

Stem	Leaf
0	1, 1, 2, 2, 3, 4
1	0, 0, 0, 1, 1, 1
2	5, 5, 7, 7, 8, 8
3	0, 1, 1, 1, 2, 2
4	0, 4, 8, 9
5	2, 6, 7, 7, 8
6	3, 6

BAR GRAPH



HISTOGRAM



OR $0<x\leq10$ $10<x\leq20$ $20<x\leq30$ $30<x\leq40$

MEASURES OF DISPERSION

Range

$$\text{range} = \text{max value} - \text{min value}$$

Note: range is greatly influenced by outliers

Interquartile range

$$\text{IQR} = Q_3 - Q_1$$

Note: spans 50% of the data set

Semi-Interquartile range

$$\text{semi} - \text{IQR} = \frac{1}{2}(Q_3 - Q_1)$$

Note: good measure of dispersion for skewed distribution

INDICATORS OF POSITION

Quartiles

The three quartiles divide the data into four quarters.
Q_1 = Lower quartile or first quartile
Q_2 = Second quartile or median
Q_3 = Upper quartile or third quartile

Percentiles

The p^{th} percentile is the value that p% of the data is less than.
Q_1= 25th percentile
Q_2= 50th percentile
Q_3= 75th percentile

eg. If the 25^{th} percentile is 12, then 25% of the data will be less or equal to 12

All other percentiles can be calculated using the formula:

$$i = \frac{p}{100}(n)$$

where;
i = the position of the p^{th} percentile

MEASURES OF CENTRAL TENDENCY FOR UNGROUPED DATA

Mean

The mean is also known as the average value. A disadvantage to the mean as a measure of central tendency is that it is highly susceptible to outliers.

$$\bar{x} = \frac{\text{sum of all values}}{\text{total number of values}} = \frac{\Sigma x}{n}$$

$\bar{x}$ = mean Σx = sum of all values
n = number of values

Mode

The mode is the value that appears most frequently in a set of data points.
Bimodal: a data set with 2 modes
Trimodal: a data set with 3 modes
* If all values have the same frequency, there is NO MODE

Median

The median is the middle number in a set of data that has been arranged in order of magnitude. The median is less affected by outliers and skewed data than the mean.

$$\text{position of median} = \frac{1}{2}(n+1)$$

Where;
n = number of values
If n = odd number, the median is part of the data set.
If n = even number, the median will be the average between the two middle numbers.

EXAMPLE:
Create a frequency table for the following ungrouped data:
1, 5, 3, 1, 2, 3, 4, 5, 1, 4, 2, 4, 4, 5, 1, 4, 2, 4, 2, 2

Streak	Frequency Chosen-Tally	Number
1	||||	4
2	~~||||~~	5
3	||	2
4	~~||||~~ |	6
5	|||	3

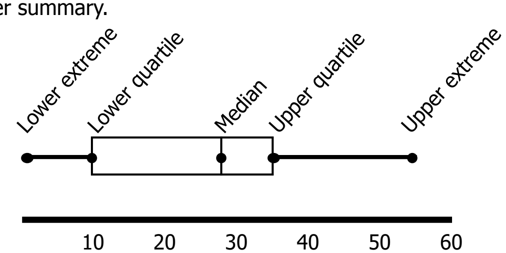
$$\text{mean} = \frac{\text{sum (frequencies} \times \text{value)}}{\text{total frequency}}$$
$$= \frac{59}{20}$$
$$= 2{,}95$$

mode : 4

$$\text{median} = \frac{1}{2}(n+1)$$
$$= \frac{1}{2}(20+1)$$
$$= 10{,}5$$
$$\therefore \text{the median is } 3$$

BOX AND WHISKER PLOT

A box and whisker plot is a visual representation of the five number summary.



FIVE NUMBER SUMMARY

1. Minimum value
2. Lower quartile Q_1
3. Median
4. Upper quartile Q_3
5. Maximum value

MEASURES OF CENTRAL TENDENCY FOR GROUPED DATA

Estimated mean

$$\text{mean}(\bar{x}) = \frac{\text{sum (frequencies} \times \text{midpoint of interval)}}{\text{total frequency (n)}}$$

$\bar{x}$ = estimated mean
n = number of values

Modal class interval

The modal class interval is the class interval that contains the greatest number of data points.

Median class interval

The median class interval is the interval that contains the middle number in a set of data points.

$$\text{position of median} = \frac{1}{2}(n+1)$$

n = number of values
If n = odd number, the median is part of the data set.
If n = even number, the median will be the average between the two middle numbers.

GROUPED DATA FREQUENCY TABLES

Class interval	Frequency (f)	Midpoint $x = \frac{\text{uper class barrier} + \text{lower class barrier}}{2}$	$(f \times x)$
$0 \leq x < 10$	3	$\frac{10+0}{2} = 5$	$3 \times 5 = 15$
$10 \leq x < 20$	7	$\frac{20+10}{15} = 15$	$7 \times 15 = 105$
$20 \leq x < 30$	4	$\frac{30+20}{2} = 25$	$4 \times 25 = 100$
total :	**14**		**220**

Mean:

$$\text{mean}(\bar{x}) = \frac{\text{sum (frequencies} \times \text{midpoint of interval)}}{\text{total frequency (n)}}$$

$$= \frac{220}{14}$$

$$= 15{,}71$$

EXAMPLE:
The mathematics marks of 200 grade 10 learners at a school can be summarised as follows:

Percentage obtained	Number of candidates
$10 \leq x < 20$	4
$20 \leq x < 30$	10
$30 \leq x < 40$	37
$40 \leq x < 50$	43
$50 \leq x < 60$	36
$60 \leq x < 70$	26
$70 \leq x < 80$	24
$80 \leq x < 90$	20

SOLUTION:

1. Calculate the approximate mean mark for the examination.

Frequency	Midpoint	$f \times x$
4	$\frac{20+10}{2} = 15$	$15 \times 4 = 60$
10	$\frac{30+20}{2} = 25$	$25 \times 10 = 250$
37	$\frac{40+30}{2} = 35$	$35 \times 37 = 1295$
43	$\frac{50+40}{2} = 45$	$45 \times 43 = 1935$
36	$\frac{60+50}{2} = 55$	$55 \times 36 = 1980$
26	$\frac{70+60}{2} = 65$	$65 \times 26 = 1690$
24	$\frac{80+70}{2} = 75$	$75 \times 24 = 1800$
20	$\frac{90+80}{2} = 85$	$85 \times 20 = 1700$
200		**10 710**

$$\text{mean}(\bar{x}) = \frac{\text{sum (frequencies} \times \text{midpoint of interval)}}{\text{total frequency (n)}}$$

$$= \frac{11\ 165}{200}$$

$$= 55{,}825$$

2. Identify the interval in which each of the following data items lies:

2.1. the median;

$$\text{median} = \frac{1}{2}(n+1)$$
$$= \frac{1}{2}(200+1)$$
$$= 100{,}5$$

median class $50 \leq x \leq 60$,
the 100^{th} value is in this class interval

2.2. the lower quartile;

Lower quartile = 25^{th} Percentile :

$$i = \frac{25}{100}(200)$$
$$i = 50$$

$\therefore$ the 25^{th} percentile is in the $30 \leq x \leq 40$ class interval

2.3. the upper quartile;

Upper quartile = 75^{th} Percentile :

$$i = \frac{75}{100}(200)$$
$$i = 150$$

$\therefore$ the 75^{th} percentile is in the $60 \leq x \leq 70$ class interval

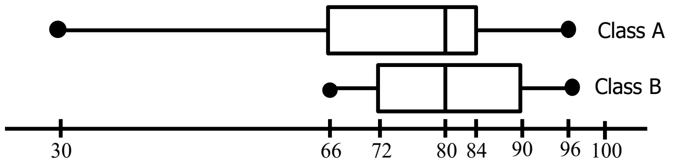
2.4. the thirteenth percentile;

$$i = \frac{13}{100}(200)$$
$$i = 26$$

$\therefore$ the 13^{th} percentile is in the $30 \leq x \leq 40$ class interval

EXAMPLE:

Examine the following box and whisker diagrams and answer the questions that follow:



1. Name the value from the five number summary that is the same for both classes.
2. For each class, explain if the data is skewed or symmetrical.

SOLUTION:

1. The median for both classes are the same.
2. Class A: skewed to the left, the data is more dispersed to the left of the median.
   Class B: skewed to the right, the data is more dispersed to the right of the median.

EXAMPLE:

The following stem and leaf diagram represents the scores of 40 people who wrote an exam.

The total of the scores is: **1544**

Stem	Leaf
1	7, 7, 8, 8, 9, 9
2	0, 2, 4, 4, 6, 7, 7
3	4, 5, 5, 5, 5, 5, 5, 5, 9, 9
4	1, 2, 2, 3, 7, 8
5	0, 3, 3, 4, 5, 7
6	3, 4, 5, 6, 6

Calculate the mean, mode and median for the information provided.

SOLUTION:

$\text{mode} = 35$

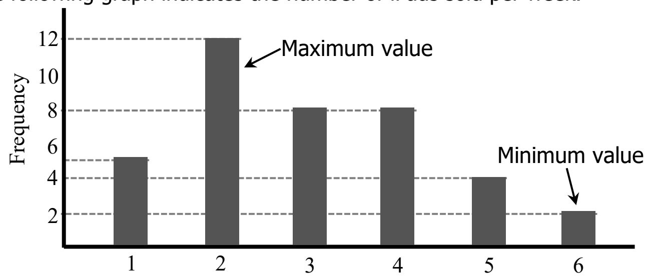
$$\begin{aligned}\text{mean} &= \frac{\text{sum of all value}}{\text{nr of values}}\\ &= \frac{1544}{40}\\ &= 38{,}6\end{aligned}$$

$$\begin{aligned}\text{position of median} &= \tfrac{1}{2}(n+1)\\ &= \tfrac{1}{2}(40+1)\\ &= 20{,}5\end{aligned}$$

$\therefore$ the median is 5

EXAMPLE:

The following graph indicates the number of iPads sold per week.



1. In which week were the sales the highest?
2. The store has a competition and the winner will be the person who bought their iPad in the middle of the sales over the 6 weeks. In which week did the winner buy their iPad?
3. That is the mean sales per week over the 6 weeks?

SOLUTION:

Weeks	Frequency
1	5
2	12
3	8
4	8
5	4
6	2
Total	**39**

1. $\text{mode} = 2$

2.
$$\begin{aligned}\text{position of median} &= \tfrac{1}{2}(n+1)\\ &= \tfrac{1}{2}(39+1)\\ &= 20\end{aligned}$$

$\therefore$ the winner bought their iPad in the 3rd week

3.
$$\begin{aligned}\text{mean} &= \frac{\text{sum of all value}}{\text{nr of values}}\\ &= \frac{39}{6}\\ &= 6{,}5\end{aligned}$$

While measures of central tendency are used to estimate "normal" values of a dataset, measures of dispersion are important for describing the spread of the data, or its variation around a central value.

Range

- Defined as the difference between the largest and smallest sample values.
- Depends only on extreme values and provides no information about how the remaining data is distributed, this means it is highly susceptible to outliers.

Interquartile Range (IQR)

- Calculated by taking the difference between the upper and lower quartiles (the 25th percentile subtracted from the 75th percentile).
- A good indicator of the spread in the center region of the data.
- More resistant to extreme values than the range.